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Optimal Control Finite Element Algorithm for Penalty Variational Formulation of the Navier-Stokes Equation*

Ma Yichen (马逸尘)

(Xi-an Jiao-tong University)

For the Stationary Navier-Stokes equations

$$\begin{cases} -\nu\Delta\vec{u}+(\vec{u}\nabla\vec{u})+\nabla p=\vec{f} & \text{in}\,\Omega\in R^3, \\ \text{div}\vec{u}=0 & \text{in}\,\Omega\in R^3, \\ \vec{u}|_{\Gamma_1}=\vec{u}_0 \quad (\nu\dfrac{\partial\vec{u}}{\partial n}-p\cdot\vec{n})|_{\Gamma_2}=\vec{g}_0. \end{cases} \tag{1}$$

we can write the penalty-variational formulation of the Eq. (1): Find $\vec{u}\in X_\varepsilon(\vec{u}_0)$ such that $\forall\ \vec{v}\in X_\varepsilon(\vec{0})$

$$a_0(\vec{u},\vec{v})+a_1(\vec{u};\vec{u},\vec{v})+\varepsilon^{-1}\cdot G(\vec{u},\vec{v})=\langle\vec{f},\vec{v}\rangle+\langle\vec{g}_0,\vec{v}\rangle|_{\Gamma_2}. \tag{2}$$

where, besides the common notations in other literatures, the following notations are also introduced. Let $\varepsilon>0$ be a parameter tending to zero and, for each ε, let

$$H^1_\varepsilon(\Omega)=\{\vec{u}:\vec{u}\in[H^1(\Omega)]^3,\ \|\vec{u}\|^2_{1,\varepsilon,\Omega}=\|\vec{u}\|^2_{1,\Omega}+\varepsilon^{-1}\cdot\|\text{div}\vec{u}\|^2_{0,\Omega}\}$$

and $X_\varepsilon(\vec{u}_0)=\{\vec{u}:\vec{u}\in H^1_\varepsilon,\ \vec{u}|_{\Gamma_1}=\vec{u}_0\}$. Moreorer, the continuous linear functionals are defined: $\forall\vec{u},\ \vec{v},\ \vec{w}\in H^1_\varepsilon(\Omega)$, $a_0(\vec{u},\vec{v})=\nu\int_\Omega\nabla\vec{u}\cdot\nabla\vec{v}dx$, $G(\vec{u},\vec{v})=\int_\Omega\text{div}\vec{u}\cdot\text{div}\vec{v}dx$, and $a_1(\vec{w};\vec{u};\vec{v})=\sum_{j=1}^{3}\int_\Omega w_j\dfrac{\partial\vec{u}}{\partial x_j}\vec{v}dx$.

An optimal control problem corresponding to the Eq. (1) is: Find $\vec{u}\in X_\varepsilon(\vec{u}_0)$ such that $\min\limits_{\vec{v}\in X_\varepsilon(\vec{u}_0)} J(\vec{v})=J(\vec{u})$ (3)

with $J(\vec{v})=\dfrac{1}{2}a_0(\vec{v}-\vec{\xi},\vec{v}-\vec{\xi})+\dfrac{1}{2\varepsilon}G(\vec{v}-\vec{\xi},\vec{v}-\vec{\xi})$, (4)

where in Eq.(4) $\vec{\xi}\in X_\varepsilon(\vec{u}_0)$ is a function of $\vec{v}$ according to the following variational problem: i.e.

$$\forall\vec{\eta}\in X_\varepsilon(\vec{0})\ \ a_0(\vec{\xi},\vec{\eta})+\varepsilon^{-1}\cdot G(\vec{\xi},\vec{\eta})=\langle\vec{F},\vec{\eta}\rangle-a_1(\vec{v};\vec{v},\vec{\eta}). \tag{5}$$

where $\langle\vec{F},\vec{\eta}\rangle=\langle\vec{f},\vec{\eta}\rangle+\langle\vec{g}_0,\vec{\eta}\rangle|_{\Gamma_2}$.

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Proposition 1 Suppose that $t\in R^1, \forall \vec{v}\in X_\varepsilon(\vec{u}_0)$, $\vec{w}\in X_\varepsilon(\vec{0})$, $\vec{\xi}$、$\vec{\xi}_1$ is the solution of Eq.(5), corresponding to $\vec{v}$ and $\vec{v}_1=\vec{v}+t\vec{w}$ respectively, then

$$\lim_{t\to 0} a_1(\vec{v};\vec{w},\vec{\xi}_1)=a_1(\vec{v};\vec{w},\vec{\xi}) \tag{6}$$

Proposition 2 The functional $J(\vec{v})$ defined by (4) and (5) has a differential operator $J'(\vec{v})\in\mathscr{L}(X_\varepsilon(\vec{0}),R)$ and $\forall\vec{w}\in X_\varepsilon(\vec{0})$, $\langle J'(\vec{v}),\vec{w}\rangle=a_1(\vec{w};\vec{v},\vec{v}-\vec{\xi})+a_1(\vec{v};\vec{w},\vec{v}-\vec{\xi})$

$$+a_0(\vec{w},\vec{v}-\vec{\xi})+\varepsilon^{-1}\cdot G(\vec{w},\vec{v}-\vec{\xi}), \tag{7}$$

where $\vec{\xi}$ is the function of $\vec{v}$ according to Eq.(5).

Proposition 3 If $\vec{v}_0$、$\vec{\xi}_0\in X_\varepsilon(\vec{u}_0)\cap[H^2(\Omega)]^3$ is a stationary point of $J(\vec{v})$, then one has

$$\begin{cases}(\vec{v}_0-\vec{\xi}_0)\nabla\vec{v}_0-\sum_{j=1}^{3}\dfrac{\partial}{\partial x_j}\left[(\vec{v}_0-\vec{\xi}_0)\vec{v}_{0j}\right]-\nu\Delta\vec{v}_0+\sum_{j=1}^{3}v_{0j}\dfrac{\partial\vec{v}_0}{\partial x_j}-\varepsilon^{-1}\nabla\cdot\mathrm{div}\vec{v}_0=\vec{f}\ \mathrm{in}\Omega,\\ \{(\vec{v}_0-\vec{\xi}_0)\vec{v}\cdot\vec{n}+(\nu\dfrac{\partial\vec{v}_0}{\partial n}+\varepsilon^{-1}\cdot\mathrm{div}\vec{v}_0\cdot\vec{n}\}|_{\Gamma_1}=\vec{g}_0,\ \ \vec{v}_0|_{\Gamma_1}=\vec{u}_0.\end{cases} \tag{8}$$

Proposition 4 A solution $\vec{v}_0$ of the variational problem (2) and $\vec{\xi}$ defined according to Eq.(5) is certainly the solution of the optimal control problem (3)～(5).

Proposition 5 Suppose that, for $\varepsilon>0$, the problem (2) has at least a solution $\vec{u}_\varepsilon$, then, for a suitable constant $K_1>0$ and the optimal control problem (3)～(5), there exists a subsequence $\{\vec{v}_n\}$ of the any minimizing sequence of $J(\vec{v})$ on the convex set $\tilde{X}_\varepsilon(\vec{u}_0)=\{\vec{u}\in X_\varepsilon(\vec{u}_0),\ \|\vec{u}\|_{1,\Omega}\leqslant K_1,\ x\in\Omega\}$ such that $\forall\ \vec{w}\in X_\varepsilon(\vec{0})$

$$\lim_{n\to\infty}\{a_0(\vec{v}_n,\vec{w})+a_1(\vec{v}_n;\vec{v}_n,\vec{w})+\varepsilon^{-1}G(\vec{v}_n,\vec{w})\}=\langle\vec{F},\vec{w}\rangle. \tag{9}$$

Proposition 6 For a suitable $K_2>0$ and on $\tilde{Y}_\varepsilon(\vec{u}_0)=\{\vec{u}\in X_\varepsilon(\vec{u}_0)\cap[H^s_\varepsilon(\Omega)]^3$, $\|\vec{u}\|_{s,\Omega}\leqslant K_2,\ x\in\Omega\}$, the optimal control probem (3)～(5) has at least a solution $\vec{v}_0$, i.e. $J(\vec{v}_0)=\min\limits_{\vec{v}\in\tilde{Y}_\varepsilon(\vec{u}_0)} J(\vec{v})$. (10)

Remark Here H^s_ε is a fractional Sobolev space for $S\in R^+$ normed by $\|u\|^2_{s,\varepsilon,\Omega}=\|u\|^2_{s,\Omega}+\varepsilon^{-1}\|\mathrm{div}\ u\|^2_{0,\Omega}$.

Proposition 7 Suppose that the hypothesis of the Proposition 5 holds, then for a suitable $K_2>0$, where $K_2=\|\vec{u}_\varepsilon\|_{S,\varepsilon,\Omega}$, the solution $\vec{v}_0$ of the optimal control problem (3)～(5) on the convex set $\tilde{Y}_\varepsilon(\vec{u}_0)$ is also the solution of the penalty-variational problem(2).

An algorithm to solve the problem (3)～(5) is given according to the conjugate gradient method recommended by R.Glowinski, i.e.:

Step o: we can take for $\vec{u}^0\in X_\varepsilon(\vec{u}_0)$ the solution of the corresponding penalty-variational problem of the Stokes Eq.:

$$\forall\vec{w}\in X_\varepsilon(\vec{0}),\ a_0(\vec{u}^0,\vec{w})+\varepsilon^{-1}\cdot G(\vec{u}^0,\vec{w})=\langle\vec{F},\vec{w}\rangle, \tag{11}$$

then compute $\vec{g}_0 \in X_\varepsilon(\vec{0})$ as the solution of the linear variational Eq.: $\forall \vec{\eta} \in X_\varepsilon(\vec{0})$,

$$a_0(\vec{g}^0, \vec{\eta}) + \varepsilon^{-1} \cdot G(\vec{g}_0, \vec{\eta}) = \langle J'(\vec{u}^0); \vec{\eta} \rangle \tag{12}$$

and set $\vec{\zeta}^0 = \vec{g}^0$. (13)

For $n \geq 0$, assuming $\vec{u}^n, \vec{g}^n, \vec{\zeta}^n$ known, compute $\vec{u}^{n+1}$, $\vec{g}^{n+1}$, $\vec{\zeta}^{n+1}$ by

Step 1: $$\lambda^n = \arg \min_{\lambda \in R} J(\vec{u}^n - \lambda \vec{\zeta}^n), \vec{u}^{n+1} = \vec{u}^n - \lambda^n \vec{\zeta}^n. \tag{14}$$

Step 2: Construction of the new descent direction. Define $\vec{g}^{n+1} \in X_\varepsilon(\vec{0})$ as the solution of the linear Eq.:

$$\forall \vec{\eta} \in X_\varepsilon(\vec{0}),\ a_0(\vec{g}^{n+1}, \vec{\eta}) + \varepsilon^{-1} \cdot G(\vec{g}^{n+1}, \vec{\eta}) = \langle J'(\vec{u}^{n+1}), \vec{\eta} \rangle, \tag{15}$$

$$\gamma^{n+1} = \frac{\int_\Omega \nabla \vec{g}^{n+1} \cdot \nabla (\vec{g}^{n+1} - \vec{g}^n) dx}{\int_\Omega |\nabla \vec{g}^n|^2 dx},$$

$$\vec{\zeta}^{n+1} = \vec{g}^{n+1} + \gamma^{n+1} \vec{\zeta}^n,$$

replace n to $n+1$, go to (14) until the accuracy is achieved.

Remark One of the characters for this method is to reduce a nonlinear variational problem (2) to finding linear variational problems (11)、(12)、(15) and a minimizing problem of the single variable (14). The fact that $J'(\vec{v})$ can be expressed in terms of integrals on Ω (see(7)) is of fundamental importance in view of finite element approximations. After completing the discretization of the problems (11)、(12)、(15) by finite element method, we obtain the stiffness matrices K, K_ε for $a_0(\vec{u}, \vec{v})$, $G(\vec{u}, \vec{v})/\varepsilon$ which are invariant in the iterative process.

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