J. Math. Res. & Exposition, 1988, No.2

Extreme points and rotundity of Orlicz-Musielak sequence spaces*

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Abstract A. Kaminska and H. Hudzik [1—10] present a series of work concerning geometry of sequence Orlicz-Musielak spaces. This paper continues their work, to give a character of extreme points of the unit balls of sequence Orlicz-Musielak spaces epuipped with Luxemburg norm. From which a criterion of rotundity is obtained immediately.

An extreme point of the unit ball of a Banach space means that it does not lie on any segment with two ends defferent from it in the unit ball.

Let X be a Banach space, N the set of all natural numbers. Let $\varphi = (\varphi_n): X \times N \to [0, +\infty]$ be a sequence of Young functions, i.e., φ_n is convex, even and $\varphi_n(0) = 0$ for every $n \in N$. Furthermore, for each $n \in N$, the following conditions are assumed (see[1]):

(*a*) $\exists$ nonzero $x \in X$ such that $\varphi_n(x) < \infty$,

(*b*) for each $x \in X$, $\varphi_n(tx): (0, +\infty) \to [0, +\infty]$ is a left-continuous function of t.

For a sequence $x = (x_n)$ of X, define $I_\varphi(x) = \sum_{n=1}^{\infty} \varphi_n(x_n)$ and

$$l_\varphi = \{x = (x_n) \subset X: \exists \lambda > 0,\ I_\varphi(\lambda x) < \infty\}$$

$$\|x\|_\varphi = \inf\{\lambda > 0: I_\varphi(x/\lambda) \leq 1\}, \quad x \in l_\varphi \tag{1}$$

then $(l_\varphi, \|\cdot\|_\varphi)$, so-called sequence Orlicz-Musielak space, is a Banach space (see[1]).

Lemma 1 For each $n \in N$ and $x \in X$, if $\varphi_n(x) < \infty$, then $\varphi_n(\lambda x)$ is a continuous, nondecrease convex funotion of $\lambda \in [0, 1]$.

Proof Take in mind the fact that φ_n is convex, nonnegative and $\varphi_n(0) = 0$.

Lemma 2 For $n \in N$, $x, y \in X$, if $\varphi_n(x) < \infty$, $\varphi_n(y) < \infty$, then φ_n is continuous on the segment $xy \overset{\text{def}}{=} \{ax + (1-a)y: a \in (0, 1)\}$.

Proof Analogously as the case that X is the real line.

Lemma 3 $\|x\|_\varphi \leq 1$ if and only if $I_\varphi(x) \leq 1$.

Proof Observe (1) and Lemma 1.

* Received Aug. 13, 1985.

Theorem 1 $x=(x_n)$ in the unit ball $U(l_\varphi)$ of l_φ is an extreme point of $U(l_\varphi)$ if and only if the following conditions are satisfied

1° $I_\varphi(x)=1$ or x_n is an extreme point of $\{w\in X: \phi_n(w)<\infty\}$ for all $n\in N$.

2° φ_n is not constant on any segment in X of which x_n is the midpoint for all $n\in N$.

3° there is at most one number $n\in N$ such that x_n is not a strictly convex point of φ_n, i.e., there exist two points y, z in X such that $x_n=\frac{1}{2}(y+z)$ and $\varphi_n(\frac{1}{2}(y+z))=\frac{1}{2}\varphi_n(y)+\frac{1}{2}\varphi_n(z)$.

Proof Necessity. Let $x=(x_n)$ be an extreme point of $U(l_\varphi)$. If 1° does not hold, then $I_\varphi(x)<1$ and there exist $n\in N$ and two points u, v in X such that $x_n=\frac{1}{2}(u+v)$ and $\varphi_n(u)<\infty$, $\varphi_n(v)<\infty$. Since φ_n is continuous on $\overline{uv}$, $\exists\, y_n,\ z_n\in\overline{uv}$ ($y_n\neq z_n$), $x_n=\frac{1}{2}(y_n+z_n)$ so closed to x_n that

$$\varphi_n(y_n)<\varphi_n(x_n)+[1-I_\varphi(x)],\quad \varphi_n(z_n)<\varphi_n(x_n)+[1-I_\varphi(x)]. \tag{2}$$

Define $y_m=z_m=x_m(m\neq n)$ and $y=(y_i)$, $z=(z_i)$, then $y\neq z$, $x=\frac{1}{2}(y+z)$ and by (2), $I_\varphi(y)=I_\varphi(x)-\varphi_n(x_n)+\varphi_n(y_n)<1$ therefore, by Lemma 3, $\|y\|_\varphi\leqslant 1$. Similarly, it is verified that $\|z\|_\varphi\leqslant 1$ contradicting the hypothesis that x is an extreme point of $U(l_\varphi)$.

If 2° is not true, then there exist $n\in N$ and two points y_n, z_n in X with $x_n=\frac{1}{2}(y_n+z_n)$ such that $\varphi_n(x_n)=\varphi_n(y_n)=\varphi_n(z_n)$. Define $y_m=z_m=x_m(m\neq n)$ and $y=(y_i)$, $z=(z_i)$, then $y\neq z$, $x=\frac{1}{2}(y+z)$ and $I_\varphi(y)=I_\varphi(z)=I_\varphi(x)<1$, also a contradiction.

If 3° fails to be satisfied, then there exist two numbers m, $n\in N$ and u_m, v_m, u_n, v_n ($u_m\neq v_m, u_n\neq v_n$) such that $x_m=\frac{1}{2}(u_m+v_m)$, $x_n=\frac{1}{2}(u_n+v_n)$ and such that

$$\varphi_m(x_m)=\frac{1}{2}\varphi_m(u_m)+\frac{1}{2}\varphi_m(v_m),\quad \varphi_n(x_n)=\frac{1}{2}\varphi_n(u_n)+\frac{1}{2}\varphi_n(v_n) \tag{3}$$

Clearly, φ_m is linear on $\overline{u_m v_m}$ and φ_n is linear on $\overline{u_n v_n}$. Without loss of generality, we may assume

$$\varphi_n(u_n)\geqslant\varphi_n(v_n),\quad \varphi_m(u_m)\leqslant\varphi_m(v_m),\quad \varphi_n(u_n)+\varphi_m(u_m)\geqslant\varphi_n(v_n)+\varphi_m(v_m) \tag{4}$$

(otherwise, exchange the places of u_n and v_n or u_m and v_m). Let

$$f(\lambda)=\varphi_n(\lambda u_n+(1-\lambda)x_n)+\varphi_m(u_m),\quad g(\lambda)=\varphi_n(\lambda v_n+(1-\lambda)x_n)+\varphi_m(v_m)$$

then $f(\lambda)$ and $g(\lambda)$ are continuous on $[0,1]$ and by (4), $f(1)\geqslant g(1)$ and $f(0)\leqslant g(0)$. Hence, there exists $\lambda_0\in[0,1]$ such that $f(\lambda_0)=g(\lambda_0)$. Denote $u_0=\lambda_0 u_n+((1-\lambda_0)x_n$, $v_0=\lambda_0 v_n+(1-\lambda_0)x_n$, then

$$\frac{1}{2}(u_0+v_0)=\lambda_0\frac{1}{2}(u_n+v_n)+(1-\lambda_0)x_n=\lambda_0 x_n+(1-\lambda_0)x_n=x_n$$

therefore, by (3)

$$\varphi_n(x_n)=\frac{1}{2}\varphi_n(u_0)+\frac{1}{2}\varphi_n(v_0) \tag{5}$$

Define $y_n=u_0$, $z_n=v_0$, $y_m=u_m$, $z_m=v_m$, $y_k=z_k=x_k(k\neq n, m)$ and $y=(y_i)$, $z=(z_i)$, then $x=\frac{1}{2}(y+z)$ and $y\neq z$ since $u_m\neq v_m$. Moreover, by (3), (5), $I_\varphi(x)=\frac{1}{2}I_\varphi(y)+\frac{1}{2}I_\varphi(z)$. Combine $f(\lambda_0)=g(\lambda_0)$, we understand $I_\varphi(y)=I_\varphi(z)=I_\varphi(x)<1$, again a contradiction.

Sufficiency. Suppose there exist $y=(y_i)$, $z=(z_i)\in U(I_\varphi)$ sucn that $x=\frac{1}{2}(y+z)$ and $y\neq z$, i.e., there exists $n\in N$ such that $y_n\neq z_n$ therefore, by 2°, $\varphi_n(y_n)\neq\varphi_n(z_n)$.

If $I_\varphi(x)\neq 1$, then by 1°, x_n is an extreme point of $\{w\in X: \varphi_n(w)<\infty\}$, therefore, $\varphi_n(y_n)+\varphi_n(z_n)=\infty$ contradicting Lemma 3. If $I_\varphi(x)=1$, then by Lemma 3 and $1=I_\varphi(x)\leqslant\frac{1}{2}I_\varphi(y)+\frac{1}{2}I_\varphi(z)$, $I_\varphi(y)=I_\varphi(z)=1$ and $\varphi_m(x_m)=\frac{1}{2}\varphi_m(y_m)+\frac{1}{2}\cdot\varphi_m(z_m)$ $(m\in N)$. It follows by 3° $x_i=y_i=z_i$ for all i in N other than n. Recall $\varphi_n(y_n)\neq\varphi_n(z_n)$, we have $I_\varphi(y)>I_\varphi(x)=1$ or $I_\varphi(z)>I_\varphi(x)=1$ contradicting the fact $y, z\in U(l_\varphi)$ in view of Lemma 3 completing the proof.

Definition We say $\varphi=(\varphi_n)$ satisfies condition Δ, if there exist $\lambda>1$, $K>1$, $a>0$ and a convergent series $\sum_{n=1}^{\infty}c_n$ such that for all large n, we have $\varphi_n(\lambda u)\leqslant K\varphi_n(u)+c_n$ for all u in X satisfying $\varphi_n(u)\leqslant a$.

For $\lambda>1$, $K>1$ and $a>0$, define

$$h_n(\lambda, K, a)=\sup\{\varphi_n(\lambda x): \varphi_n(\lambda x)\geqslant K\varphi_n(x),\ \varphi_n(x)\leqslant a, x\in X\} \qquad (6)$$

Where we provide $\sup_{t\in E} t=\infty$ when set E is unbounded.

Theorem 2 The following are equivalent

(Ⅰ) φ does not satisfy condition Δ,

(Ⅱ) for any $\lambda>1$, $K>1$, $a>0$ and $m\in N$, $\sum_{n=m}^{\infty}h_n(\lambda, K, a)=\infty$,

(Ⅲ) there exists $x=(x_n)$ in l_φ such that $\|(0,\cdots,0,x_m,x_{m+1},\cdots)\|_\varphi=1$ for all $m\in N$.

Proof (Ⅰ)⇒(Ⅱ). If (Ⅱ) is not true, then there exist $\lambda>1$, $K>1$, $a>0$ and $m\in N$ such that $\sum_{n=m}^{\infty}h_n(\lambda, K, a)<\infty$. Given $n>m$ and $u\in X$ satisfying $\varphi_n(u)\leqslant a$, if $\varphi_n(\lambda u)>h_n(\lambda, K, a)$, then by (6), $\varphi_n(\lambda u)\leqslant K\varphi_n(u)+h_n(\lambda, K, a)$. This inequality of cause holds when $\varphi_n(\lambda u)\leqslant h_n(\lambda, K, a)$ contradicting (Ⅰ).

(Ⅱ)⇒(Ⅲ). Since $\sum_{n=1}^{\infty}h_n(2,2^2,\frac{1}{2^2})=\infty$, there exists $N_1\geqslant 1$ such that $\sum_{n=1}^{N_1}h_n(2,2^2,\frac{1}{2^2})>1$ and $\sum_{n=1}^{N_1-1}h_n(2,2^2,\frac{1}{2^2})\leqslant 1$ (where $N_1=1$ may happen, in this case, we always provide $\sum_{n=i}^{i-1}a_n=0$). Similarly, since $\sum_{n=N_1+1}^{\infty}h_n(1+\frac{1}{2},2^3,\frac{1}{2^3})=\infty$, there exists $N_2\geqslant N_1+1$ such that $\sum_{n=N_1+1}^{N_2}h_n(1+\frac{1}{2},2^3,\frac{1}{2^3})>1$ and $\sum_{n=N_1+1}^{N_2-1}h(1+\frac{1}{2},2^3,\frac{1}{2^3})\leqslant 1$

In the same way, we may choose $N_3 \geqslant N_2+1$ such that $\sum_{n=N_2+1}^{N_3} h_n(1+\frac{1}{2}, 2^4, \frac{1}{2^4}) > 1$ and $\sum_{n=N_1+1}^{N_2-1} h_n(1+\frac{1}{3}, 2^4, \frac{1}{2^4}) < 1, \cdots$. It follows by (6) there exists $x_n \in X$ such that $\varphi_n(x_n) \leqslant 1/2^{i+2}$, $\varphi_n((1+\frac{1}{i+1})x_n) \leqslant 2^{i+2}\varphi_n(x_n)$ and

$$\sum_{n=N_i+1}^{N_{i+1}} \varphi_n((1+\frac{1}{i+1})x_n) > 1, \quad \sum_{n=N_i+1}^{N_{i+1}-1} \varphi_n((1+\frac{1}{i+1})x_n) \leqslant 1 \qquad (7)$$

for all n in N satisfying $N_i+1 \leqslant n \leqslant N_{i+1}$ and all $i=0,1,2,\cdots$ where $N_0=0$.

Defining $x=(x_n)$, by (7) and $\varphi_{N_{i+1}}(x_{N_{i+1}}) \leqslant 1/2^{i+2}$, we have

$$I_\varphi(x) = \sum_{i=0}^{\infty}\Big[\sum_{n=N_i+1}^{N_{i+1}} \varphi_n(x_n)\Big] \leqslant \sum_{i=0}^{\infty}\Big[\sum_{n=N_i+1}^{N_{i+1}} \frac{1}{2^{i+2}}\varphi_n((1+\frac{1}{i+1})x_n) + \varphi_{N_{i+1}}(x_{N_{i+1}})\Big]$$

$$\leqslant \sum_{i=0}^{\infty}\Big[\frac{1}{2^{i+2}} + \frac{1}{2^{i+2}}\Big] = 1$$

Therefore, $\|x\|_\varphi \leqslant 1$. On the other hand, for any $\lambda > 1$ and m in N, there exists $i_0 \geqslant m$ such that $1+1/(i_0+1) < \lambda$, it follows by (7),

$$\sum_{n=m}^{\infty}\varphi_n(\lambda x_n) \geqslant \sum_{n=N_{i_0}+1}^{\infty} \varphi_n(\lambda x_n) \geqslant \sum_{i=i_0}^{\infty} \sum_{n=N_i+1}^{N_{i+1}} \varphi_n((1+\frac{1}{i+1})x_n) \geqslant \sum_{i=i_0}^{\infty} 1 = \infty$$

Recall (1), we have $\|(0,\cdots,0,x_m,x_{m+1},\cdots)\|_\varphi = 1$.

(Ⅲ) ⇒ (Ⅰ). If φ satisfies condition Δ, then there exist $\lambda > 1$, $K > 1$, $a > 0$, $N_1 \in$ N and $c_n \geqslant 0$ ($n \in$ N) with $\sum_{n=1}^{\infty} c_n < \infty$ such that $\varphi_n(\lambda u) \leqslant K\varphi_n(u) + c_n$ for all u in X satisfying $\varphi_n(u) \leqslant a$ and all $n \geqslant N_1$

Given $x=(x_n)$ in l_φ, choose $m \geqslant N_1$ such that $\sum_{n=m}^{\infty} c_n < \frac{1}{2}$, $K\sum_{n=m}^{\infty}\varphi_n(x_n) < \min(\frac{1}{2}, a)$, then

$$\sum_{n=m}^{\infty}\varphi_n(\lambda x_n) \leqslant \sum_{n=m}^{\infty}\big[K\varphi_n(x_n) + c_n\big] < \frac{1}{2} + \frac{1}{2} = 1$$

By (1), $\|(0,\cdots,0,x_m,x_{m+1},\cdots)\|_\varphi \leqslant \frac{1}{\lambda} < 1$ contradicting (Ⅲ).

Theorem 3 l_φ is rotund iff the following conditions are provided

(i) $\sup\{\lambda: \varphi_n(\lambda u) < \infty\} < 1$ for all nonzero u in X with $\varphi_n(u) < 1$ and all n in N,

(ii) φ satisfies condition Δ,

(iii) φ_n is not constant on any segment in $\{w \in X: \varphi_n(w) \leqslant 1\}$ for all n in N,

(iv) For any two points i, j in N and each (u,v) in $\{(x,y): \varphi_i(x) + \varphi_j(y) \leqslant 1, x, y \in X\}$, u is a strictly convex point of φ_i or v is a strictly convex point of φ_j.

Prooff Necessity. If (i) is not true, then there exists i in N and x_i in X such that $\varphi_i(x_i) < 1$ and $\varphi_i(\lambda x) = \infty$ for all $\lambda > 1$. Define $x_n = 0$ ($n \neq i$) and $x = (x_n)$, then by (1), $\|x\|_\varphi=1$. On the other hand, for any j in N other than i, by the definition of φ_j, there exists nonzero u in X such that $\varphi_j(-u)=\varphi_j(u)<\infty$, therefore, $x_j=0=\frac{1}{2}u+\frac{1}{2}(-u)$ is not an extreme point of $\{w\in X: \varphi_j(w)<\infty\}$. Combine $I_\varphi(x)=\varphi_i(x_i)<1$ with 1° in Theorem 1, x is not an extreme point of $U(l_\varphi)$, a contradiction.

If (ii) does not hold, by (Ⅲ) in Theorem 2, there exists $m>1$ and $x=(x_n)$ in l_φ such that $\sum_{n=m}^{\infty}\varphi_n(x_n)<1$ and $\|(0,\cdots,0,x_m,x_{m+1},\cdots)\|_\varphi=1$. Analogously as in the case (i), it is easy to verify that x is not an extreme point of $U(l_\varphi)$.

If (iii) does not hold, then there exist i in N and two points u, v in $\{w\in X: \varphi_i(w)<1\}$ such that φ_i is constant on $\overline{uv}$. For any j in different from i and nonzero x' in X, let $\lambda_0=\sup\{\lambda>0: \varphi_j(\lambda x')+\varphi_i(\frac{u+v}{2})<1\}$ and define $x_i=\frac{1}{2}(u+v)$, $x_j=\lambda_0 x'$, $x_k=0$ $(k\neq i,j)$ and $x=(x_n)$ then by the definition of λ_0 and (1), it is easily verified that $\|x\|_\varphi=1$ and by 2° in Theorem 1, that x is not an extreme point of $U(l_\varphi)$.

If (iv) does not hold, then there exist two points i, j in N and x_i, x_j in X such that $\varphi_i(x_i)+\varphi_j(x_j)<1$. Choose k in N other than i, j and nonzero w in X let $\lambda_0=\sup\{\lambda>0: \varphi_i(x_i)+\varphi_j(x_j)+\varphi_k(\lambda w)<1\}$ and define $x_k=\lambda_0 w$, $x_m=0$ $(m\neq i,j,k)$, $x=(x_n)$, then it is similarly verified that x norms 1 not being an extreme point of $U(I_\varphi)$.

Sufficiency. For given $x=(x_n)\in l_\varphi$ with $\|x\|_\varphi=1$, we have to show that x is an extreme point of $U(l_\varphi)$ which is equivalent to verify 1°, 2°, 3° in Theorem 1.

1°. we show $I_\varphi(x)=1$. By (ii), there exist $\lambda>1, a>0$ $m\in$ N and $c_n>0$ $(n\in \mathrm{N})$ with $\sum_{n=1}^{\infty}c_n<\infty$ such that $\varphi_n(\lambda u)<K\varphi_n(u)+c_n$ whenever $n\geq m$, u in X with $\varphi_n(u)\leq a$. If $I_\varphi(x)<1$, then there exists $N_1\in N$, $N_1\geq m$ such that

$$\sum_{n=N_1}^{\infty}\left[K\varphi_n(x_n)+c_n\right]<\frac{1}{2}\left[1-I_\varphi(x)\right] \qquad \sum_{n=N_1}^{\infty}\varphi_n(x_n)\leq a$$

for each n in N, since $\varphi_n(x_n)\leq I_\varphi(x)<1$, by condition (i), there exists $\lambda_n>1$ such that $\varphi_n(\lambda_n x_n)<\infty$ therefore, $\sum_{n=1}^{N_1-1}\varphi_n(\lambda x_n)$ is a continuous function of $\lambda\in[0, \min_{n<N_1}\lambda_n]$. Since $\min_{n<N_1}\lambda_n>1$, there exists $\lambda_0>1$ such that $\sum_{n=1}^{N_1-1}\varphi_n(\lambda_0 x_n)<\sum_{n=1}^{N_1-1}\varphi_n(x_n)+\frac{1}{2}\left[1-I_\varphi(x)\right]$. Define $\lambda^*=\min(\lambda',\lambda_0)$, then

$$I_\varphi(\lambda^* x)\leq\sum_{n=1}^{N_1-1}\varphi_n(\lambda_0 x_n)+\sum_{n=N_1}^{\infty}\varphi_n(\lambda' x_n)<\sum_{n=1}^{N_1-1}\varphi_n(x_n)+\frac{1}{2}\left[1-I_\varphi(x)\right]$$

$$+\sum_{n=N_1}^{\infty}\left[K\varphi_n(x_n)+c_n\right]<\sum_{n=1}^{N_1-1}\varphi_n(x_n)+\left[1-I_\varphi(x)\right]\leqslant 1$$

Hence, $\|x\|_\varphi<\frac{1}{\lambda^*}<1$ contradicting $\|x\|_\varphi=1$.

2°. For each n in N, since $\varphi_n(x_n)\leqslant 1$, by (iii), φ_n is not a constant on any segment of which x_n is the midpoint.

3°. If there exists some i in N such that x_i is not a strictly convex point, then for any j in N different from i, by (iv) and $\varphi_i(x_i)+\varphi_j(x_j)\leqslant 1$, x_j is a strictly convex point of φ_j.

Combining 1°, 2°, 3° proves the theorem.

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