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Uniqueness of Best L Approximation For Continuous Functions*

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This paper deals with the problem of uniqueness of best L approximation for continuous functions. In this paper we use a new method to establish a kind of characterization theorems and a sufficient condition for unicity space-an n-dimensional subspace of C[a,b], from which every function f in C[a,b] has a unique best L approximation.

I. This paper discusses the problem of uniqueness of best L approximation for continuous functions.

Setting the notation, X will denote an interval [a,b], and C(X) will be the space of all real-valued continuous functions f on X with the L norm

$\|f\| = \int_x |f| \int_x |f(x)|\,dx$ writ

$z(f)=\{x\in X:\ f(x)=0\}$, $z_+(f)=\{x\in X:\ f(x)>0\}$, $z_-(f)=\{x\in X:\ f(x)<0\}$.

Let V be an n-dimensional subspace of C(X). V is said to be an unicity space if every function f in C(X) has an unique best L approximation.

Strauss [1, 2] and others have given characterizations for unicity spaces. Using a new method we are going to establish another kind of characterization theorems (Section Ⅱ) and a sufficient condition (Section Ⅲ) for unicity spaces.

II. In [1], Strauss defined by H_v the set of all functions in C(X) such that for every h in H_V there exists a v in V satisfying $|h(x)|=|v(x)|$ for $x\in X$. We want to describe it using another method.

Definition I. Let $v\in V$. (Z_+, Z_-) is said to be an H-partition to v if it satisfies that

(a) $Z_+\cup Z_-=X\setminus Z(v)$;

(b) Z_+ and Z_- are open subsets in relation to X, i.e., for every $x\in Z_+$ (corresp. Z_-) there exists a $t>0$ such that $(x-t,\ x+t)\cap X\subset Z_+$ (corresp. Z_-);

(c) $Z_+\cap Z_-=\phi$.

The following lemma points out the relation between an H-partition to $v\in V$

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and a function h in the set H_V corresesoonding to v.

Lemma 1. Let $v \in V$. (Z_+, Z_-) is an H-partition to v, if and only if, there exists an h in H_V such that

$$Z_+ = Z_+(h), \quad Z_- = Z_-(h) \text{ and } |h| = |v|.$$

Proof. Let (Z_+, Z_-) be an H-partition to v. Set

$$h(x) = \begin{cases} |v(x)|, & x \in Z_+ \\ -|v(x)|, & x \in Z_- \\ 0, & x \in Z(v). \end{cases}$$

Of course, we have that $Z_+ = Z_+(h)$, $Z_- = Z_-(h)$ and $|h| = |v|$. Therefore to prove $h \in H_V$ it only suffices to show that $h \in C(X)$.

Since $h(x) = |v(x)|$ on Z_+ and Z_+ is open in relation to X, $h(x)$ is continuous on Z_+.

The same conclusion is also true for $h(x)$ on Z_-.

Let $t \in Z(v)$. It means that $h(t) = v(t) = 0$. Whence,

$$|h(t+\Delta t) - h(t)| = |h(t+\Delta t)| = |v(t+\Delta t)| \to |v(t)| = 0$$

as $\Delta t \to 0$. So $h(x)$ is also continuous at t.

Conversely, suppose that $h \in H_V$ satisfying $|h| = |v|$ and $Z_+ = Z_+(h)$, $Z_- = Z_-(h)$. It is easy to see that (a), (b) and (c) in the definition are satisfied. Thus (Z_+, Z_-) is an H-partition to v.

Now using this lemma we can establish the main result of the section.

Theorem 2. V is an unicity space, if, and only if, there exises no nontrivial function v^* in V and H-partition (Z_+, Z_-) to v^* such that

$$\left| \int_{Z_+} v - \int_{Z_-} v \right| \leqslant \int_{Z(v^*)} |v|, \quad \forall v \in V. \tag{1}$$

Proof. Theorem 3 in [1] says that V is a unicity space, if and only if, there exists no nontrivial function h in H_V satisfying

$$\left| \int_X v \operatorname{sgn} h \right| \leqslant \int_{Z(h)} |v|, \quad \forall v \in V. \tag{2}$$

Now, by definition, for every h in H_V there exists a v^* in V such that $|h| = |v^*|$. Of course, We have $Z(h) = Z(v^*)$. Lemma 1 claims that (Z_+, Z_-) is an H-partition to v^*, where

$$Z_+ = Z_+(h) \text{ and } Z_- = Z_-(h).$$

Conversly, Lemma 1 also claims that for every v^* in V and every H-partition (Z_+, Z_-) to v^*, there exists an $h \in H_V$ such that $Z_+ = Z_+(h)$, $Z_- = Z_-(h)$ and $|h| = |v^*|$, the last one of which implies that $Z(h) = Z(v^*)$. Therefore, there exists no nontrivial function h in H_V satisfying (2), if and only if, there exists no nontrivial function v^* in V and H-partition (Z_+, Z_-) to v^* satisfying (1).

This proves our theorem.

Definition 2. (Z, Z_+, Z_-) is said to be a generalized H-partition to v in V if it satisfies that

(a) $X = Z \cup Z_+ \cup Z_-$;

(b) Z is a closed subset and Z_+, Z_- are open subsets in relation to X;

(c) $Z \cap Z_+ = Z \cap Z_- = Z_+ \cap Z_- = \emptyset$;

(d) $Z \subset Z(v)$. (3)

Theorem 3. V is a unicity space, if and only if, there exists no nontrivial function v^* in V and generalized H-partition (Z, Z_+, Z_-) to v^* such that

$$\left| \int_{Z_+} v - \int_{Z_-} v \right| < \int_{Z} |v|, \quad \forall v \in V. \tag{4}$$

Proof. It is easy to see that if (Z_+, Z_-) is an H-partition to v^*, then (Z, Z_+, Z_-) is a generalized H-partition to v^*, where $Z = Z(v^*)$. Thus the sufficicncy of the theorom follows directly from Theorem 2.

For the necessity of the theorem suppose, to the contrary, that there exists a nontrivial v^* in V and a generalized H-partition (Z, Z_+, Z_-) to v^* satisfying (4). Let $Z^* = Z(v^*)$, $Z_+^* = Z_+ \setminus Z^*$ and $Z_-^* = Z_- \setminus Z^*$. Obviously, (Z_+^*, Z_-^*) is an H-partition to v^*. Moreover, from (3) and (4) it follows that for any $v \in V$

$$\int_{Z(v^*)} |v| = \int_{Z^*} |v| = \int_{Z} |v| + \int_{Z_+ \cap Z^*} |v| + \int_{Z_- \cap Z^*} |v| \geqslant \left| \int_{Z_+} v - \int_{Z} v \right|_{+}^{+} \int_{Z_+ \cap Z^*} |v| + \int_{Z_- \cap Z^*} |v|$$

$$\geqslant \left| \int_{Z_+} v - \int_{Z_+ \cap Z^*} v - \int_{Z_-} v + \int_{Z_- \cap Z^*} v \right| = \left| \int_{Z_+ \setminus Z^*} v - \int_{Z_- \setminus Z^*} v \right| = \left| \int_{Z_+^*} v - \int_{Z_-^*} v \right|.$$

This shows that (1) is satisfied. By Theorem 2, V is not a unicity space, a contradiction.

This completes the proof of the theorem.

From Theorem 2 and Theorem 3 we can easily obtain the following corollaries.

Corollary 4. Let every nontrivial function v in V be nonzero almost everywhere. Then V is an unicity space, if and only if, there exists no nontrivial function v^* in V and H-partition (Z_+, Z_-) to v^* such that

$$\int_{Z_-} v = \int_{Z_-} v, \quad \forall v \in V$$

Corollary 5. Let every nontrivial function v in V have only a number of finite zeros. Then V is a unicity space, if and only if, there exists no nontrivial v^* in V and k points,

$$a = x_0 < x_1 < x_2 < \cdots < x_k < x_{k+1} = b, \tag{5}$$

such that

$$v^*(x_i) = 0, \quad i = 1, 2, \cdots, k \tag{6}$$

and

$$\sum_{i=0}^{k}(-1)^i \int_{X_i}^{X_{i+1}} v = 0, \quad \forall v \in V. \tag{7}$$

Proof. Sufficiency. Suppose on the contrary that V is not an unicity space. By Theorem 2 there exists a nontrivial function $v^* \in V$ and an H-partition (Z_+, Z_-) to v^* such that (1) holds. By Definition 1 we have $\overline{Z}_+ \cap \overline{Z}_- \subset Z(v^*)$. under the assumptions of the corollary we can suppose that $\overline{Z}_+ \cap \overline{Z}_- = \{x_1, x_2, \cdots, x_k\}$ and, furthermore, suppose that (5) is satisfied. Then in this case (1) becomes (7) and (6) is valid. This is a contradiction.

Necessity. If not and suppose that there exists a nontrivial function v^* in V and k points (5) satisfying (6) and (7). Denote

$$I_0 = [x_0, x_1), I_1 = (x_1, x_2), \cdots, I_{k-1} = (x_{k-1}, x_k), I_k = (x_k, x_{k+1}],$$

$$Z = \{x_1, x_2, \cdots, x_k\}, \quad Z_+ = \bigcup_{j < \frac{1}{2}k} I_{2j}, \quad Z_- = \bigcup_{j < \frac{1}{2}(k-1)} I_{2j+1}.$$

Then (Z, Z_+, Z_-) must be a generalized H-partition to v^*. Moreover, (4) follows from (7). Applying Theorem 3, V is not an unicity space, a contradiction.

This proves the corollary.

Ⅲ. This section provides a sufficient condition insuring L-uniqueness which is as follows.

Condition H. There exists, for every nontrivial v in V and every H-partition (Z_+, Z_-) to v, a nontrivial function w in V such that the following holds:

$$w(x) \begin{cases} = 0 \text{ on } Z(v) \text{ almost everywhere}, \\ \geq 0, \quad x \in Z_+, \\ \leq 0, \quad x \in Z_-. \end{cases} \tag{8}$$

Now we state the following

Theorem 6. If V satisfies Condition H, then V is an unicity space.

Proof. Let v, (Z_+, Z_-) and w be defined as in Condition H. Since w satisfies (8),

$$\left| \int_{Z_+} w - \int_{Z_-} w \right| = \int_{Z_+ \cup Z_-} |w| > 0 \text{ and } \int_{Z(v)} |w| = 0.$$

This means that m does not satisfy (1). By Theorem 2, V is a unicity space.

To conclude this section we present an equivalent condition to Condition H.

Theorem 7. Condition H is equivalent to the condition: For every nontrivial h in H_V, there exists a nontrivial function w in V such that

$$wh \geq 0 \tag{9}$$

and

$$w(x) = 0 \text{ on } Z(h) \text{ almost everywhere.} \tag{10}$$

Proof. $(\Rightarrow)$. By the definition of H_V, for a nontrivial $h \in H_V$ there exists a nontrivial v in V such that $|h| = |v|$. Put

$$Z_+ = Z_+(h), \quad Z_- = Z_-(h). \tag{11}$$

By Lemma 1, (Z_+, Z_-) is an H-partition to v. By Condition H, there exists a nontrivial w in V satisfying (8). Thus (9) and (10) follow from (8) and (11).

($\Leftarrow$). Lemma 1 tells us that for a nontrivial $v \in V$ and an H-partition (Z_+, Z_-) to v there exists a nontrivial $h \in H_V$ such that

$$Z_+ = Z_+(h), \quad Z_- = Z_-(h) \text{ and } |h| = |v|.$$

By the assumptions of the theorem there exists a nontrivial w in V satisfying (9) and (10). From (9), (10) and the above relations, (8) easily follows. This means that Condition H is satisfied.

Remark. In comparison with Condition A in [2] the following assumption, which is necessary there, is avoided in Condition H: every function v in V has only a finite number of separated zeros.

References

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(from 266)

(a) $X_0 = \phi$; (b) $p \bar{\in} G$; (c) $\|F(\cdot, p)\| > \|F^*\|$. Moreover, if p is a minimum to F, then each of them implies that p is the unique minimum to F.

Theorem 3 Let $p \in K$. If $\overline{f_1} < \overline{f_2}$ and $\overline{f_1} \le f^- \le f^+ \le \overline{f_2}$, then p is a unique minimum to F in K if and only if F possesses a generalized alternation system.

Theorem 4 Let $F(x, y)$ satisfy Assumptions (A) and (C), and $f^-, f^+ \in C(X)$. Suppose that for each x, $F(x, y)$ is convex with respect to y. If F has a unique minimum in the L_∞norm, then F has a unique minimum in the L_1 norm.

References

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