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A Strong Maximum Principle for a Family of Singular Hyperbolic Operators*

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I. Introduction

The maximum principle for the Laplace operator states that if a smooth function u satisfies the differential inequality

$$\Delta u \geq 0$$

at each point of a domain D, then u attains its maximum on ∂D, the boundary of D. The maximum principle for the heat operator states that if a smooth function u satisfies the differential inequality

$$\Delta u - \frac{\partial u}{\partial t} \geq 0$$

at each point of a cylinder $R = \{(x, t) \mid x \in D, 0 < t \leq T\}$, where D is a domain of x-space, then the maximum of u on the closure $R \cup \partial R$ must occur on the portion of the boundary of R which is either at the bottom of R or along the sides $\partial D \times [0, T]$ [1], [6].

The strong maximum principles corresponding to the above principles state as follows [6], [10]:

The strong maximum principle for the Laplace operator: If a smooth function u satisfies

$$\Delta u \geq 0$$

at each point of a domain D, and attains its maximum U at an interior point of D, then $u \equiv U$ in D;

The strong maximum principle for the heat operator: Suppose the inequality

$$\Delta u - \frac{\partial u}{\partial t} \geq 0 \quad \text{in R}$$

holds, where $R = D \times (0, T]$ is a cylinder in (x, t)-space, D a domain of x-space. If the maximum U of u is attained at any point (x_0, t_0) of R, then

$$u(x, t) = U \quad \text{for } x \in \overline{D}, \ 0 \leq t \leq t_0.$$

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Also, the strong maximum priciples for general uniformly elliptic operators and uniformly parabolic ones hold[3], [5], [6]. For some kind of elliptic systems and parabolic systems, strong maximum principle in a sense is valid. However, for other kind of those, only maximum principle holds[2], [6-9].

Although strong maximum principles hold for both uniformly elliptic and uniformly parabolic operators, but there is some essential difference between them. In [4], the author has studied some maximum principles for a family of singular hyperbolic operators. Naturally, one may ask if strong maximum principle in a sense holds for these operators? If, of course, there is such a strong maximum principle, yet we can't expect that it possesses some special property, that as in the case of parabolic operators, or even as in the case of elliptic ones, because of their different instrinsic quality.

We begin with some examples.

2. Some examples

In this paper, the object to be studied is a family of singular hyperbolic operators in a domain E:

$$L_{p,q,c} = \frac{\partial^2}{\partial x^2} - (h(x))^2 \frac{\partial^2}{\partial t^2} + q\frac{h'(x)}{h(x)}\frac{\partial}{\partial x} + ph'(x)\frac{\partial}{\partial t} + c(x,t),$$

where p and q are real parameters, functions $h(x)$ and $c(x,t)$ satisfy, respectively

$$h \in C^2(0, M] \cap C^1[0, M]; \quad h(0) = 0; \quad h'(x) > 0 \text{ for } x > 0, \tag{2.1}$$

$$c \in C^0(\overline{E} \setminus C), \quad c \leq 0, \tag{2.2}$$

the domain $E = \{(x, t) \mid x > 0,\ t - H(x) > 0,\ t + H(x) < 2H(M)\}$, in which $H(x) = \int_0^x h(s)\,ds$, $M > 0$ and $C = \partial E \cap \{(x, t) \mid x = 0\}$. Denote

$$\Gamma_1 = \partial E \cap \{(x, t) \mid t - H(x) = 0,\ x > 0\},$$
$$\Gamma_2 = \partial E \cap \{(x, t) \mid t + H(x) = 2H(M),\ 0 < x < M\}.$$

We have the following maximum principle[4]:

Theorem A Suppose that (a) p, q satisfy

$$p - q - 1 \leq 0, \tag{2.3}$$

$$4h^2c + (p - q - 1)[2hh'' + (p + q - 3)(h')^2] \geq 0 \text{ in } \overline{E} \setminus C, \tag{2.4}$$

and that (b) a function $u \in C^2(\overline{E} \setminus C) \cap C^1(\overline{E})$ satisfies

$$L_{p,q,c}u \geq 0 \quad \text{in } E, \tag{2.5}$$

$$u, \text{ as a function of } x, \text{ decreases on } \Gamma_1, \tag{2.6}$$

$$\max_{\overline{E}} u \geq 0 \quad \text{if } c \not\equiv 0. \tag{2.7}$$

Then we have

$$\max_{C} u = \max_{\overline{E}} u. \tag{2.8}$$

Moreover, (2.8) holds without the requirement (2.7) if $c\equiv 0$ (in particular, if $p-q-1=0$).

Conversely, (2.8) is violated if (2.3) doesn't hold even though all the remaining conditions are satisfied.

In the following examples, we always suppose that $f\in C^2[0, 2H(M))$, $\max\limits_{[0, 2H(M)]} f(s)=0$, and that (2.3), (2.4) hold.

Example 1 Let $u=f(t-H(x))$, where $f'(s)\leq 0$ for $0\leq s\leq 2H(M)$. Then we have

$$L_{p,q,c}u=(p-q-1)h'(x)f'(t-H(x))+c(x, t)f(t-H(x))\geq 0 \text{ in E. If}$$

$$f(s)=\begin{cases} 0 & \text{for } 0\leq s\leq H(M) \\ -\exp\left(-\dfrac{1}{s-H(M)}\right) & \text{for } H(M)<s\leq 2H(M), \end{cases}$$

then u satisfies all of the above conditions and attains its maximum zero on the characteristic curve Γ: $t-H(x)=H(M)$.

We observe that the point $\left(\frac{M}{2}, H(M)+H\left(\frac{M}{2}\right)\right)\in E$ (because $h'>0$) is on the characteristic Γ and that $u\left(\frac{M}{2}, H(M)+H\left(\frac{M}{2}\right)\right)=0=\max\limits_{\bar{E}} u$. Also, we observe that $u\equiv 0$ on the curve $\left\{(x, t)\in\bar{E}\,\middle|\, t+H(x)=H(M)+2H\left(\frac{M}{2}\right), x\geq\frac{M}{2}\right\}$ through the point $\left(\frac{M}{2}, H(M)+H\left(\frac{M}{2}\right)\right)$.

Example 2 Suppose that the strict inequality in (2.3) holds. We consider the function $u=f(t-kH(x))$, where k is a constant, $0<k<1$, such that $p-kq-k<0$. It always is possible to select k in this way because $p-q-1<0$. If $f''\leq 0$, $f'\leq 0$, then a computation shows that

$$\begin{aligned} L_{p,q,c}u=&(k^2-1)(h(x))^2f''(t-kH(x))+(p-kq-k)f'(t-kH(x)) \\ &+c(x, t)f(t-kH(x))\geq 0 \text{ in E.} \end{aligned}$$

If

$$f(s)=\begin{cases} 0 & \text{for } 0\leq s\leq (1-k)H(M) \\ -\exp\left(-\dfrac{4H(M)}{s-(1-k)H(M)}\right) & \text{for } (1-k)H(M)<s\leq 2H(M), \end{cases}$$

then f fulfills all of the above conditions, and u attains its maximum zero on the curve $t-kH(x)=(1-k)H(M)$.

We observe that $u=0=\max\limits_{\bar{E}} u$ at the point $P=\left(\frac{M}{2}, (1-k)H(M)+kH\left(\frac{M}{2}\right)\right)\in E$. However, $u\not\equiv 0$ on the characteristic curve $t-H(x)=(1-k)\left(H(M)-H\left(\frac{M}{2}\right)\right)$ which contains the point P. In particular, $u\not\equiv 0$ in $E_k\equiv E\cap$

$\{(x,t)\,|\,t-H(x)<(1-k)(H(M)-H(\frac{M}{2}))\}$. However, $u\equiv 0$ on the curve $\{(x,t)\in\overline{E}\,|\,t+H(x)=(1-k)H(M)+(1+k)H(\frac{M}{2}),\ x\geq\frac{M}{2}\}$ through the point $(\frac{M}{2},\ (1-k)H(M)+kH(\frac{M}{2}))$.

Example 3 suppose $p+q+1\geqslant 0$ with holding of the strict inequality in (2.3). Letting $u=f(t+H(x))$, where $f(s)=-(s-2H(M))^2$ for $0\leqslant s\leqslant 2H(M)$, we have

$$L_{p,q,c}u=(p+q+1)h'(x)f'(t+H(x))+c(x,t)f(t+H(x))\geqslant 0$$

because $f\leqslant 0$, $f'\geqslant 0$, $p+q+1\geqslant 0$, $h'\geqslant 0$ and $c\leqslant 0$. We notice that u attains its maximum zero only on the characteristic curve $t+H(x)=2H(M)$.

These examples give us a stimulus to form the following strong maximum principle for our singular hyperbolic operators.

3. Strong maximum principle

In order to state our results couveniently, we make some notations. Suppose that P is a point which belongs to $E\cup\Gamma_2$. Construct the characteristic curve from P which is in the same family of characteristic curves with Γ_2 and ends with P^* on Γ_1. We denote by Γ_{PP^*} this characteristic curve PP^* (if $P\in\Gamma_2$, this segment of characteristic curve is just a portion of Γ_2).

Theorem 1 (Strong maximum principle for singular hyperbolic operators). Suppose that p, q satisfy

$$p-q-1<0, \tag{2.3'}$$

$$4h^2c+(p-q-1)[2hh''+(p+q-3)(h')^2]>0 \quad \text{in } \overline{E}\setminus C, \tag{2.4'}$$

and that $u\in C^2(\overline{E}\setminus C)\cap C^1(\overline{E})$ satisfies (2.5)—(2.7). If the maximum U of u is attained at a point P of $E\cup\Gamma_2$ and if $U\geq 0$, then

$$u\equiv U \quad \text{on } \Gamma_{PP^*}. \tag{3.1}$$

In the case of $c\equiv 0$, (3.1) holds without the requirement $U\geq 0$.

Proof Suppose $u(P)=U\equiv\max_{\overline{E}} u\geq 0$. If $P\in E$, then we have $(u_x+h(x)u_t)(P)=0$; if $P\in\Gamma_2$, then $(u_x+h(x)u_t)(P)\geq 0$. In a word, we have

$$(u_x+h(x)u_t)(P)\geq 0. \tag{3.2}$$

The following identity holds for the operator $L_{p,q,c}$

$$D_-[(h(x))^aD_+v]=(h(x))^aL_{p,q,c}v-D_-(Av)+[A'-(h(x))^ac]v,$$
$$\text{for } v\in C^2(\overline{E}\setminus C),\ \text{in } \overline{E}\setminus C, \tag{3.3}$$

where $D_\pm=\frac{\partial}{\partial x}\pm h(x)\frac{\partial}{\partial t}$, $a=(p+q-1)/2$, $A=((p-q+1)/2)h'(x)(h(x))^{a-1}$, A' is the derivative of A with respect to x. Integrate the identity (3.3), in which

v is replaced by u, with respect to x along the characteristic Γ_{PP^*}. We find

$$(h(x))^a D_+ u \Big|_P^{P^*} \geq (-Au)\Big|_P^{P^*} + \int_{\Gamma_{PP^*}} (A' - h^a c) u \,dx$$

$$= (-Au)\Big|_P^{P^*} + \int_{\Gamma_{PP^*}} (h^a c - A')(u(P) - u)\,dx - u(P)\int_{\Gamma_{PP^*}} h^a c\,dx$$

$$+ u(P) A \Big|_P^{P^*}$$

$$= A(P^*)(u(P) - u(P^*)) + \int_{\Gamma_{PP^*}} (h^a c - A')(u(P) - u)\,dx - u(P)\int_{\Gamma_{PP^*}} h^a c\,dx. \tag{3.4}$$

Because $A(P^*) > 0$ (by $(2.3)'$ and (2.1)), $u(P) = U \equiv \max_{\overline{E}} u \geq 0$, $h^a c - A' = \frac{1}{4} h^{\frac{p+q-5}{2}}$. $\{4h^2 c + (p-q-1)[2hh'' + (p+q-3)(h')^2]\} > 0$ on Γ_{PP^*} (by $(2.4)'$) and $c \leq 0$, hence each term of the right side of (3.4) is nonnegative, and we obtain $(h(x))^a D_+ u \Big|_P^{P^*} \geq 0$. If the conclusion of the theorem were false, i.e., $u \not\equiv U$ on Γ_{PP^*}, then the second term of the right side of (3.4) would be positive, therefore $(h(x))^a D_+ u \Big|_P^{P^*} > 0$, that is to say (by the condition (2.6))

$$(D_+ u)(P) < \left(\frac{h(P^*)}{h(P)}\right)^a (D_+ u)(P^*) \leq 0. \tag{3.5}$$

This would contradict (3.2) and thereby (3.1) is proved.

It is obvious from (3.4) that we can remove the restriction $U \geq 0$ when $c \equiv 0$. The proof is complete now.

Remark 1 We see from the examples 1—3 that Theorem 1 is not nominal, i.e., there do exist such functions satisfying all conditions of Theorem 1. and that we can't expect the of point

$$S = \{(x, t) \in \overline{E} \setminus C \quad u(P) = U\}$$

is bigger that Γ_{pp^*} when all the condition of Theorem 1 are filfulled.

Remark 2 Also we see from these examples that if $(2.3)'$, $(2.4)'$ hold, there exists a nonconstant function $u \in C^2(\overline{E} \setminus C) \cap C^1(\overline{E})$ which satisfies the conditions (2.5)—(2.7) and attains its maximum on C as well in $E \cup \Gamma_2$. To go a step further, for any given point $P \in E \cup \Gamma_2$, a nonconstant function $u \in C^2(\overline{E} \setminus C) \cap C^1(\overline{E})$ can be found which satisfies the conditions (2.5)—(2.7) and attains its maximum at the point P. For example, the function $u = f(t - H(x))$ is just what we need, where

$$f(s) = \begin{cases} 0 & \text{for } 0 \leq s \leq t_P - H(x_P) \\ -\exp\left(-\dfrac{1}{s - (t_P - H(x_P))}\right) & \text{for } t_P - H(x_P) < s \leq 2H(M), \end{cases}$$

and (x_P, t_P) is the coordinate of the point P.

Remark 3 Clearly the theorem A is a consequence of the theorem 1 when $(2.3)'$, $(2.4)'$ hold, and this is why we call the theorem 1 a strong maximum principle.

Theorem 2 In the case of $p-q-1=0$, $c\equiv 0$, for any given point $P\in E\cup\Gamma_2$, there exists a function $u\in C^2(\overline{E}\setminus C)\cap C^1(\overline{E})$ satisfying (2.5)—(2.6) such that $u(P)=U\equiv\max\limits_{\overline{E}} u$ and $u<U$ on $\Gamma_{PP^*}\setminus\{P\}$.

Proof It is not difficult to verify that the function $u(x, t)=-[(t-H(x))-(t_P-H(x_P))]^2$ is just what we need.

Remark 4 The theorem 2 points out that the operator $L_{p,q,c}$ satisfies, if $p-q-1=0$, $c\equiv 0$, only the maximum principle in the theorem A, but not the strong maximum principle in the sense of the theorem 1.

From the derivation of the inequality (3.4) and (3.4) itself, we can draw several corollaries.

Corollary 1 Suppose that the conditions (2.3)—(2.7) hold and that there exists a point $P\in E\cup\Gamma_2$ such that $u(P)=U\equiv\max\limits_{\overline{E}} u$. Then we have $L_{p,q,c}u\equiv 0$ on Γ_{PP^*}. Besides these, if $U>0$, then $c\equiv 0$ on Γ_{PP^*}.

In fact, if $L_{p,q,c}u\not\equiv 0$ on Γ_{PP^*}, it would follow from (2.5) and (3.4) that $(h(x))^a D_+u\,|_P^{P^*}>0$, i.e., (3.5) holds. This would be contrary to (3.2). Moreover, if $c\not\equiv 0$ on Γ_{PP^*} and $U>0$, we would obtain, from (2.2) and (3.4), $(h(x))^a D_+u\,|_P^{P^*}>0$.

Before stating the others, we make the following notations. For $0\le s<2H(M)$, $0\le r_1<r_2\le 2H(M)$, set

$$\Gamma^{(1)}_{s,r_1,r_2}=\{(x,t)\in\overline{E}\setminus C\,|\,t-H(x)=s,\ r_1<t+H(x)<r_2\},$$
$$\Gamma^{(1)}_{s,r_1}=\{(x,t)\in\overline{E}\setminus C\,|\,t-H(x)=s,\ r_1<t+H(x)\le 2H(M)\},$$
$$\Gamma^{(2)}_{r_j}=\{(x,t)\in E\cup\Gamma_2\,|\,t+H(x)=r_j\},\ j=1,2.$$

Let E_{s,r_1,r_2} be domain bounded by $\Gamma^{(1)}_{s,r_1,r_2}$, $\Gamma^{(2)}_{r_1}$, $\Gamma^{(2)}_{r_2}$ and C. The domain E_{s,r_1} is bounded by $\Gamma^{(1)}_{s,r_1}$, $\Gamma^{(2)}_{r_1}$, Γ_2 and C.

Corollary 2 Suppose that (2.3)—(2.7) hold and that there are three real numbers s, r_1, r_2, $0\le s<2H(M)$, $0\le r_1<r_2\le 2H(M)$, such that $L_{p,q,c}u\not\equiv 0$ on $\Gamma^{(1)}_{s,r_1,r_2}$. Then the maximum of u can't be attained on the domain E_{s,r_1,r_2}. If $L_{p,q,c}u\not\equiv 0$ on $\Gamma^{(1)}_{s,r_1}$, then u can't attain its maximum on $E_{s,r_1}\cup(\Gamma_2\cap\partial E_{s,r_1})$. In particular, if $L_{p,q,c}u\not\equiv 0$ on Γ_1, then u can attain its maximum only on C. When $c\equiv 0$, we have the result without the requirement that the maximum of u be nonnegative.

Corollary 3 Suppose that (2.3)—(2.4) hold and that there are three real numbers s, r_1, r_2, $0\le s<2H(M)$, $0\le r_1<r_2\le 2H(M)$, such that $c\not\equiv 0$ on $\Gamma^{(1)}_{s,r_1,r_2}$. Then any function $u\in C^2(\overline{E}\setminus C)\cap C^1(\overline{E})$ satisfying (2.5),(2.6) and having a positive maximum $U>0$ can't attain its maximum on E_{s,r_1,r_2}. If $c\not\equiv 0$ on $\Gamma^{(1)}_{s,r_1}$, such a function u can't attain its maximum on $E_{s,r_1}\cup(\Gamma_2\cap\partial E_{s,r_1})$. In particular, if $c\not\equiv 0$ on Γ_1, such a function u can attain its maximum only on C.

Corollary 4 Suppose that (2.3)′, (2.4)—(2.7) hold. Suppose that there is a point $Q=(x_Q, t_Q)\in\Gamma_1$ such that $u(R)=U\equiv\max_E u$ for $R\in\Gamma_1$, $t_R\leq t_Q$ and $u(R)<U$ for $R\in\Gamma_1$, $t_R>t_Q$. Denote $r=t_Q+H(x_Q)$. Then u can't attain its maximum on $E_{0;r}\cup\Gamma_2$. In particular, if $u<U$ on Γ_1, then u can't attain its maximum on $E\cup\Gamma_2$. If $c\equiv 0$, the result holds without the requirement that max u be nonnegative.

Remark 5 If, instead of the condition (2.6), we have

$$u\text{, as function of } x\text{, decreases strictly on } \Gamma_1, \qquad (2.6)''$$

or

$$D_x u \quad 0 \qquad \text{on } \Gamma_1. \qquad (2.6)'''$$

then it must occur that u U on Γ_1. Therefore it follows that

Corollary 5 Suppose (2.3), (2.4), (2.5), 2.6)'' (or (2.6)'''), (2.7) hold. Then u can attain its maximum only on C. If $c\equiv 0$, the result holds without the requirement that max u be nonnegative.

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(from 126) In particular, no point of E 's isolated.

Corollary 2 Let X be an Π_1 space and let A: X→X be bounded conti nuous accretive map, F: X→X ball-condensing and C: X→X compact. Supp Suppose that $\lambda I-F-C+A$ is locally one-to-one for $\lambda\geq 1$. Let $T=F+C-A$. Th Then T has a fixed point in X if and only if $\liminf_{\lambda\downarrow\lambda_0}\|x_\lambda\|<\infty$ where $\lambda_0=\inf E$. Moreover, if $Tx\neq x$ for all x X, then each eigenvector x E of T lies in an unbounded component of E. In particular, no point of E is isolated.

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