Journal of Mathematical Research and Exposition
Vol.9, No.1, Feb., 1989

Global Solutions for Reaction Diffusion System Satisfying Nonlinear Boundary Conditions

Zhang Gongan

(Department of Mathematics, Jilin University, Changchun)

§ 1 Introduction

In recent year, the reaction diffusion system has aroused the great interest of many authors. Applying semigroup methed Franz Rothe [1] discussed problem (2.1)—(2.3) with first and third boundary conditions. Paul, D. Norman proved the existence of the solutions of problem (2.1)—(2.3)($M=0$) with linear boun dary conditions [2], [3]. In this paper, we investigate the existence-uniqueness of the solutions of coupled system (2.1)—(2.3) with nonlinear boundary condition.

The outline of this paper as follows: In section 2, we state the assumptions and prove the comparison-existence theorem. In seetion 3, we prove the uniqu-eness of the solution of problem (2.1)—(2.3). In section 4, we discuss the large time behaviors of the solutions.

§ 2 Existence and Comparison Theorem of Coupled System

We consider the following coupled reaction-diffusion system

$$\begin{cases} u_{it} = f_i(t, x, u_1, u_2, \cdots, u_N), & i = 1, \cdots, M, \\ u_{it} - \mathcal{L}_i u_i = f_i(t, x, u_1, u_2, \cdots, u_N), & i = M+1, \cdots, N. \end{cases} \quad t \in (0, T], x \in \Omega \quad (2.1)$$

together with the boundary and initial conditions

$$B_i[u_i] = a_i(x)\frac{\partial u_i}{\partial v} - g_i(t, x, u_1, \cdots, u_N) = h(t, x), \quad i = M+1, \cdots, N, \quad t \in (0, T], \ x \in \partial\Omega \quad (2.2)$$

$$u_i(0, x) = \varphi_i(x), \qquad i = 1, \cdots, N, \qquad x \in \Omega, \quad (2.3)$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in $\mathbf{R}^n$ $a_i(x) > 0$ are continuous functions on $\partial\Omega$, $\frac{\partial}{\partial v}$ is the outward nomal (or conomal) derivative on $\partial\Omega$ and $\mathcal{L}_i$ $(i = M+1, \cdots, N)$ are uniformly elliptic operators in the form

$$\mathcal{L}_i = \sum_{j,k}^{n} \frac{\partial}{\partial x_j}(a_{j,k}^{(i)}(x)\frac{\partial}{\partial x_k}) + \sum_{j=1}^{n} a_j^{(i)}(x)\frac{\partial}{\partial x_j}, \quad i = M+1, \cdots, N, \quad (2.4)$$

and there are constants $a_i > 0$ such that

* *Received Sept.* 15, 1986.

$$\sum_{j,k=1}^{n} a_{j,k}^{(i)}(x)\xi_j\xi_k \geq a_i \sum_{j=1}^{n} \xi_j^2, \tag{2.5}$$

for all x $\overline{\Omega}$ and n-dimensional real vector $\xi=(\xi_1,\cdots,\xi_n)$.

We denote $v_i(t,x)=(u_{1_i},u_{2_i},\cdots,u_{k_i})$, $w_i(t,x)=(u_{1_i'},u_{2_i'},\cdots,u_{k_i'})$ $(k_i+k_i'=N-1)$, $V_i(t,x)=(u_{i_1},u_{i_2},\cdots,u_{i_k})$, $W_i(t,x)=(u_{i_1'},u_{i_2'},\cdots,u_{i_k'})$ $(i_k+i_k'=N-1)$ and $u_i(t,x)$ as a component of the vector $(u_1,u_2,\cdots,u_N)$ does not appear in the vectors $v_i(t,x)$, $w_i(t,x)$, $V_i(t,x)$ and $W_i(t,x)$. We rewrite the boundary value problem (2.1)—(2.3) as the following form:

$$\begin{cases} u_{it}=f_i(t,x,u_i;v_i;w_i), & i=1,\cdots,M, \\ u_{it}-\mathscr{L}_i u_i=f_i(t,x,u_i;v_i;w_i), & i=M+1,\cdots,N, \end{cases} \quad t\in(0,T],\ x\in\Omega, \tag{2.1}$$

$$B_i[u_i]=\frac{\partial u_i}{\partial \nu}-g_i(t,x,u_i;V_i;W_i)=h_i(t,x).\quad i=M+1,\cdots,N,\quad t\in(0,T],\ x\in\partial\Omega, \tag{2.2}$$

$$u_i(0,x)=\varphi_i(x), \qquad i=1,\cdots,N \qquad x\in\Omega. \tag{2.3}$$

Assume that for each $i=M+1,\cdots,N$, the coefficients of the operator $\mathscr{L}_i$ are smooth on Ω; $h_i(t,x)$, $\varphi_i(x)$ are smooth functions; f_i is a Hölder continuous function in $\mathbf{R}^+\times\Omega\times\mathbf{R}^N$, where $\mathbf{R}^+=[0,\infty)$. $\varphi_i(x)$ and $h_i(t,x)$ satisfy the compatibility condition at $t=0$. For convenience, we set $D_T=\Omega\times(0,T]$, $\overline{D}_T=\overline{\Omega}\times[0,T]$, $\Gamma_T=\partial\Omega\times(0,T]$, where $T<\infty$, but can be arbitriary large.

Definition 1 For each $i=1,\cdots,N$, the function $f_i(t,x,u_i;v_i;w_i)$ is said to be quasimonotone nondecreasing (resp. nonincreasing) in a subset S of $\mathbf{R}^n$, if $f_i(t,x,u_i;v_i;w_i)$ is monotone nondecreasing (resp nonincreasing) in $w_i=(u_{1_i'},u_{2_i'},\cdots,u_{k_i'})$ and there exists a constant $M_i\geq 0$ such that $M_iu_i+f_i(t,x,u_i;v_i;w_i)$ is monotone nonincreasing in $(u_i;v_i)=(u_i;u_{1_i},u_{2_i},\cdots,u_{k_i})$.

The definition of monotone property of the boundary function $g_i(t,x,u_i;V_i;W_i)$ are similar.

Definition Let $\widetilde{U}=(\widetilde{u}_1,\widetilde{u}_2,\cdots,\widetilde{u}_N)$, $\underset{\sim}{U}=(\underset{\sim}{u}_1,\underset{\sim}{u}_2,\cdots,\underset{\sim}{u}_N)$ be smooth functions with value in a bounded subset of $\mathbf{R}^N$, $\widetilde{U}$, $\underset{\sim}{U}$ are called upper and lower solution of the problem (2.1)—(2.3) respectively, if the following inequalities hold

$$\begin{cases} \widetilde{u}_{it}-f_i(t,x,\widetilde{u}_i;\widetilde{v}_i;\underset{\sim}{w}_i)\geq 0\geq \underset{\sim}{u}_{it}-f_i(t,x,\underset{\sim}{u}_i;\underset{\sim}{v}_i;\widetilde{w}_i), & i=1,\cdots,M, \quad (t,x)\in D_T \\ \widetilde{u}_{it}-\mathscr{L}_i\widetilde{u}_i-f(t,x,\widetilde{u}_i;\widetilde{v}_i;\underset{\sim}{w}_i)\geq 0\geq \underset{\sim}{u}_{it}-\mathscr{L}_i\underset{\sim}{u}_i-f_i(t,x,\underset{\sim}{u}_i;\underset{\sim}{v}_i;\widetilde{w}_i), & \\ \qquad i=M+1,\cdots,N \quad (t,x)\in D, \end{cases} \tag{2.6}$$

$$\frac{\partial\widetilde{u}_i}{\partial\nu}-g_i(t,x,\widetilde{u}_i;\widetilde{V}_i;\underset{\sim}{W}_i)\geq h_i(t,x)\geq\frac{\partial\underset{\sim}{u}_i}{\partial\nu}-g_i(t,x,\underset{\sim}{u}_i;\underset{\sim}{V}_i;\widetilde{W}_i),\quad i=M+1,\cdots,N, \tag{2.7}$$

$$\widetilde{u}_i(0,x)\geq\varphi_i(x)\geq\underset{\sim}{u}_i(0,x), \qquad i=1,\cdots,N,\ x\in\Omega. \tag{2.8}$$

For given the reaction function $f_i(t,x,u_i;v_i;w_i)$ and the boundary function $g_i(t,x,u_i;V_i;W_i)$ and a subset S in $\mathbf{R}^N$, we assume that there exists an order pair of upper and lower solutions $\widetilde{U}=(\widetilde{u}_1,\widetilde{u}_2,\cdots,\widetilde{u}_N)$, $\underset{\sim}{U}=(\underset{\sim}{u}_1,\underset{\sim}{u}_2,\cdots,\underset{\sim}{u}_N)$ and definite

$$S(D_T)=\{(u_1,u_2,\cdots,u_N),\ u_i\in C(\overline{D}_T),\ \underset{\sim}{u}_i(t,x)\le u_i(t,x)\le \tilde{u}_i(t,x),\ i=1,\cdots,N\}. \tag{2.9}$$

If $S(D_T)$ is contained in S, then it suffices to take $S=S(D_T)$.

In order to ensure the uniqueness, we also assume that there exists constants c_i, C_i such that for each $i=1,2,\cdots,N$

$$|f_i(t,x,u_1,u_2,\cdots,u_N)-f_i(t,x,\hat{u}_1,\hat{u}_2,\cdots,\hat{u}_N)|\le c_i\sum_{j=1}^{N}|u_j-\hat{u}_j|, \tag{2.10}$$

$$|g_i(t,x,u_1,u_2,\cdots,u_N)-g_i(t,x,\hat{u}_1,\hat{u}_2,\cdots,\hat{u}_N)|\le C_i\sum_{j=1}^{N}|u_j-\hat{u}_j|, \tag{2.11}$$

$u=(u_1,u_2,\cdots,u_N)$, $\hat{u}=(\hat{u}_1,\hat{u}_2,\cdots,\hat{u}_N)\in S(D_T)$.

To establish an existence comparison theorem in terms of upper and lower solutions, we consider the sequence $\{U^{(k)}\}=\{(u_1^{(k)},\cdots,u_N^{(k)})\}$ $(k=1,2,\cdots)$ obtained from the linear system

$$\begin{cases} u_{it}^{(k)}+M_i u_i^{(k)}=M_i u_i^{k-1}+f_i(t,x,u_i^{(k-1)};v_i^{(k-1)};w_i^{(k-1)}),\ i=1,\cdots,M. \\ u_{it}^{(k)}-\mathscr{L}_i u_i^{(k)}+M_i u_i^{(k)}=M_i u_i^{(k-1)}+f_i(t,x,u_i^{(k-1)};v_i^{(k-1)};w_i^{(k-1)}). \\ \qquad i=M+1,\cdots,N,\quad (t,x)\in D_T, \end{cases} \tag{2.12}$$

$$\frac{\partial u_i^{(k)}}{\partial v}+K_i u_i^{(k)}=K_i u_i^{(k-1)}+G_i(t,x,u_i^{(k-1)};V_i^{(k-1)};W_i^{(k-1)}),\ i=M+1,\cdots,N,\quad (t,x)\in\Gamma_T, \tag{2.13}$$

$$u_i^{(k)}(0,x)=\varphi_i(x),\qquad i=1,\cdots,N,\quad x\in\Omega, \tag{2.14}$$

$k=1,2,\cdots$, where $G_i(t,x,u_i;V_i;W_i)=g_j(t,x;u_i;V_i;W_i)+h_i(t,x)$, K_i is a constant (as M_i in definition 1) in the definition of the quasimonotone property of the boundary function $g_i(t,x,u_i;V_i;W_i)$. For each k, the above system consists of N linear uncoupled initial problem of the ordinary differential equations and initial boundary value problem of the partial differential equations and therefore the existence of $\{(u_1^{(k)},u_2^{(k)},\cdots,u_N^{(k)})\}$ fallows from the standard existence theorem of the ordinary differential equation and partial differential equation. To ensure that $\{(u_1^{(k)},u_2^{(k)},\cdots,u_N^{(k)})\}$ is a monotone sequence and it converges to an unique solution of the problem (2.1)—(2.3), it's necessary to choose a proper initial iteration. By using $(\tilde{u}_1,\tilde{u}_2,\cdots,\tilde{u}_N)$ and $(\underset{\sim}{u}_1,\underset{\sim}{u}_2,\cdots,\underset{\sim}{u}_N)$ as two distinct initial iteration we can construct two sequences $\{\overline{U}^{(k)}\}=\{(\overline{u_1}^{(k)},\overline{u_2}^{(k)},\cdots,\overline{u_N}^{(k)})\}$, $\{\underline{U}^{(k)}\}=\{(\underline{u}^{(k)},\cdots,u_N^{(k)})\}$ obtained from the linear system

$$\begin{cases} \overline{u}_{it}^{(k)}+M_i\overline{u}_i^{(k)}=M_i\overline{u}_i^{(k-1)}+f_i(t,x,\overline{u}_i^{(k-1)};\overline{v}_i^{(k-1)},\underline{w}_i^{(k-1)}),\ i=1,\cdots,M,\ (t,x)\in D_T, \\ \overline{u}_{it}^{(k)}-\ _i\overline{u}_i^{(k)}+M_i\overline{u}_i^{(k)}=M_i\overline{u}_i^{(k-1)}+f_i(t,x,\overline{u}_i^{(k-1)};\overline{v}_i^{(k-1)};\underline{w}_i^{(k-1)}), \\ \qquad i=M+1,\cdots,N; \end{cases} \tag{2.15}$$

$$\frac{\partial\overline{u}_i^{(k)}}{\partial v}+k_i\overline{u}^{(k)}=K_i\overline{u}^{(k-1)}+G_i(t,x,\overline{u}_i^{(k-1)};\overline{V}_i^{(k-1)};\underline{W}^{(k-1)}),\ i=M+1,\cdots,N,\quad (t,x)\in\Gamma_T, \tag{2.16}$$

$$\overline{u}_i^{(k)}(0,x)=\varphi_i(x),\qquad i=1,\cdots,N,\quad x\in\Omega; \tag{2.17}$$

and

$$\begin{cases} \underline{u}_{it}^{(k)} + M_i \underline{u}_i^{(k)} = M_i \underline{u}_i^{(k-1)} + f_i(t, x, \underline{u}_i^{(k-1)}; \underline{v}_i^{(k-1)}; \overline{w}_i^{(k-1)}), \ i = 1, \cdots, M, \\ \underline{u}_{it}^{(k)} - \mathscr{L}_i u_i^{(k)} + M_i u_i^{(k)} = M_i \underline{u}_i^{(k-1)} + f_i(t, x, \underline{u}_i^{(k-1)}; \underline{v}_i^{(k-1)}; \overline{w}_i^{(k-1)}), \ i = M+1, \cdots, N, \end{cases}$$

$(t, x) \in D_T,$ (2.18)

$$\frac{\partial \underline{u}_i^{(k)}}{\partial v} + K_i \underline{u}_i^{(k)} = K_i \underline{u}_i^{(k-1)} + G_i(t, x, \underline{u}_i^{(k-1)}; \underline{V}_i^{(k-1)}, \overline{W}_i^{(k-1)}),$$

$$i = M+1, \cdots, N, \ (t,x) \in \Gamma_T, \quad (2.19)$$

$$u_i^{(k)}(0, x) = \varphi_i(x), \qquad i = 1, \cdots, N, \ x \in \Omega, \quad (2.20)$$

respectively.

Lemma 2.1 Let $\tilde{U}, \underset{\sim}{U}$ be a pair of upper and lower solutions of the problem (2.1)—(2.3). Then the maximum sequence $\{\bar{U}^{(k)}\}$ is monotone nonincreasing and the minimal sequence $\{\underline{U}^{(k)}\}$ is monotone nondecreasing. Moreover,

$$\underset{\sim}{u}_i \le \underline{u}_i^{(1)} \le \cdots \le \underline{u}_i^{(k)} \le \underline{u}_i^{(k+1)} \le \cdots \le \bar{u}_i^{(k+1)} \le \bar{u}_i^{(k)} \le \cdots \le \bar{u}_i^{(1)} \le \tilde{u}_i, \ i = 1, \cdots, N, \quad (2.21)$$

Proof From (2.12)—(2.14) and (2.6)—(2.8) we obtain

$$\begin{cases} \theta_{it} + M_i \theta_i = M_i \underset{\sim}{u}_i + f_i(t, x, \underset{\sim}{u}_i; \underset{\sim}{v}_i; \tilde{w}_i) - (\underset{\sim}{u}_{it} + M_i \underset{\sim}{u}_i) \ge 0, i = 1, \cdots, M, (t,x) \in D_T, \\ \theta_{it} - \mathscr{L}_i u_i + M_i \theta_i = M_i \underset{\sim}{u}_i + f_i(t, x, \underset{\sim}{u}_i; \underset{\sim}{v}_i; \tilde{w}_i) - [\underset{\sim}{u}_{it} - \mathscr{L}_i \underset{\sim}{u}_i + M_i \underset{\sim}{u}_i) \ge 0, \end{cases} \quad (2.22)$$

$$i = M+1, \cdots, N, \ (t, x) \in D_T,$$

$$\frac{\partial \theta_i}{\partial v} + K_i \theta_i = K_i \underset{\sim}{u}_i + G_i(t, x, \underset{\sim}{u}_i; \underset{\sim}{V}_i; \tilde{W}_i) - (\frac{\partial \underset{\sim}{u}_i}{\partial v} + K_i \underset{\sim}{u}_i) \ge 0,$$

$$i = M+1, \cdots, N, (t, x) \in \Gamma_T, \quad (2.23)$$

$$\theta_i(0, x) = \varphi_i(x) - u_i(0, x) \ge 0, \qquad i = 1, \cdots, N, x \in \Omega, \quad (2.24)$$

where $\theta_i = \underline{u}_i^{(1)} - \underline{u}_i^{(0)} \equiv \underline{u}_i^{(1)} - \tilde{u}_i$. The maximum principle shows that $\theta_i(t, x) \ge 0$, so that $\underset{\sim}{u}_i \le \bar{u}_i^{(1)}$. Similarly, we prove $\bar{u}_i^{(1)} \le \tilde{u}_i$. we show that $\underline{u}_i^{(1)} \le \bar{u}_i^{(1)}$. It follows from (2.15)—(2.20) and the definition of quasimonotone of f_i, G_i that

$$\theta_{it} + M_i \theta_i = M_i(\tilde{u}_i - \underset{\sim}{u}_i) + [f_i(t, x, \tilde{u}_i; \tilde{v}_i; \underset{\sim}{w}_i) - f_i(t, x, \underset{\sim}{u}_i; \tilde{v}_i; \underset{\sim}{w}_i)]$$

$$+ \sum_{j_i} [f_i(t, x, \underset{\sim}{u}_i, \cdots, \tilde{u}_{j_i}, \cdots, \underset{\sim}{w}_i) - f_i(t, x, \underset{\sim}{u}_i, \cdots, \underset{\sim}{u}_{j_i}, \cdots; \underset{\sim}{w}_i)]$$

$$+ \sum_{j_i'} [f_i(t, x, \underset{\sim}{u}_i; \underset{\sim}{v}_i; \cdots, \underset{\sim}{w}_{j_i'}, \cdots) - f_i(t, x, \underset{\sim}{u}_i; \underset{\sim}{v}_i; \cdots, \tilde{w}_{ji}, \cdots)],$$

$i = 1, \cdots, M, (t, x) \in D_T$; Similarly,

$$\theta_{it} - \mathscr{L}_i \theta_i + M_i \theta_i = M_i(\tilde{u}_i - \underset{\sim}{u}_i) + [f_i(t, x, \tilde{u}_i; \tilde{v}_i; \underset{\sim}{w}_i) - f_i(t, x, \underset{\sim}{u}_i; \underset{\sim}{v}_i; \tilde{w}_i)] \ge 0,$$

$$i = M+1, \cdots, N, \quad (t, x) \in D_T,$$

$$\frac{\partial \theta_i}{\partial v} + K_i \theta_i = K_i(\tilde{u}_i - \underset{\sim}{u}_i) + [G_i(t, x, \tilde{u}_i; \tilde{V}_i; \underset{\sim}{W}_i) - G_i(t, x, \underset{\sim}{u}_i; \tilde{V}_i; \underset{\sim}{W}_i)]$$

$$+ \sum_{i_i} [G_i(t, x, \underset{\sim}{u}_i; \cdots, \tilde{u}_{i_i}, \cdots; \underset{\sim}{W}_i) - G_i(t, x, \underset{\sim}{u}_i; \cdots, \underset{\sim}{u}_{i_i}, \cdots; \underset{\sim}{W}_i)]$$

$$+ \sum_{i_i'} [G_i(t, x, \underset{\sim}{u}_i; V_i; \cdots u_{i_i'}, \cdots) - G_i(t, x, \underset{\sim}{u}_i; \underset{\sim}{V}_i; \cdots, \tilde{u}_{i_i'}, \cdots)] \ge 0,$$

$$i = M+1, \cdots, N, \ (t, x) \in \Gamma_T,$$

$$\theta_i(0, x) = 0,$$

where $\theta_i = \bar{u}_i^{(1)} - \underline{u}_i^{(1)}$, the quantily in the brackets represent the variation of the functions f_i or G_i which fallows from the variation of one component u_i (or u_{i_j}, u_{j_i} etc) of the vector $(u_1, u_2, \cdots, u_N)$ only, then by maximum principl we prove $\theta_i \geq 0$, i.e. $\cdots$

$$\underset{\sim}{u_i} \leq \underline{u}_i^{(1)} \leq \overline{u}^{(1)} \leq \widetilde{u}_i \quad i = 1, 2, \cdots, N. \tag{2.25}$$

We assume, by induction, that $\underline{u}_i^{(k-1)} \leq \underline{u}_i^{(k)} \leq \overline{u}_i^{(k)} \leq \overline{u}_i^{(k-1)}$, $i = 1, \cdots, N$, $k = 1, 2, \cdots, k_0$. By the same argument as in the proof of the relation (2.25), we get $\underline{u}_i^{(k_0)} \leq \underline{u}_i^{(k_0+1)} \leq \overline{u}_i^{(k_0+1)} \leq u_i^{(k_0)}$, $i = 1, \cdots, N$. This completes the proof of Lemma.

We now state the main result in this paper as the following:

Theorem 2.2 Let $\widetilde{U} = (\widetilde{u}_1, \widetilde{u}_2, \cdots, \widetilde{u}_N)$, $\underset{\sim}{U} = (\underset{\sim}{u}_1, \underset{\sim}{u}_2, \cdots, \underset{\sim}{u}_N)$ be a pair upper and lower solutions of the problem (2.1)—(2.3) for $f_i(t, x, u_1, u_2, \cdots, u_N)(i = 1, 2, \cdots, N)$, $G_i(t, x, u_1, u_2, \cdots, u_N)(i = M+1, \cdots, N)$ on $S(D_T)$. Suppose that conditions (2.10), (2.11) hold, then the sequences $\{\overline{U}^{(k)}\} = \{(\overline{u}_1^{(k)}, \overline{u}_2^{(k)}, \cdots, \overline{u}_N^{(k)})\}$ and $\{\underline{U}^{(k)}\} = \{\underline{u}_1^{(k)}, \underline{u}_2^{(k)}, \cdots, \underline{u}_N^{(k)}\}$ converge monotone from above and below, respectively, to a unique solution of the problem (2.1)—(2.3).

Proof In view of Lemma 2.1 the pointwise limit

$$\lim_{k\to\infty} \overline{u}_i^{(k)}(t, x) = \overline{u}_i(t, x), \quad \lim_{k\to\infty} \underline{u}_i^{(k)}(t, x) = \underline{u}_i(t, x), \ (t, x) \in \overline{D}_T, \tag{2.26}$$

exist and

$$\underset{\sim}{u_i}(t, x) \leq \underline{u}_i(t, x) \leq \overline{u}_i(t, x) \leq \widetilde{u}_i(t, x), \quad (t, x) \in \overline{D}_T, \ i = 1, 2, \cdots, N. \tag{2.27}$$

For $M = 0$, $N = 2$, paper [3],[2] has proved that the limit function $(\overline{u}_1, \overline{u}_2, \cdots, \overline{u}_N)$, $(\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_N)$ is a solution of the problem

$$\begin{cases} \overline{u}_{it} = f_i(t, x, \overline{u}_i; \overline{v}_i; \underline{w}_i), & i = 1, \cdots, M, \\ \overline{u}_{it} - \mathscr{L}_i \overline{u}_i = f_i(t, x, \overline{u}_i; \overline{v}_i; \underline{w}_i), & i = M+1, \cdots, N, \end{cases} \quad (t, x) \in D_T, \tag{2.28}$$

$$\frac{\partial \overline{u}_i}{\partial \nu} = G_i(t, x, \overline{u}_i; \overline{V}_i; \underline{W}_i), \quad i = M+1, \cdots, N, \quad (t, x) \in \Gamma_T \tag{2.29}$$

$$\overline{u}_i(0, x) = \varphi_i(x), \qquad i = 1, \cdots, N, \ x \in \Omega, \tag{2.30}$$

and

$$\begin{cases} \underline{u}_{it} = f_i(t, x, \underline{u}_i; \underline{v}_i; \overline{w}_i), & i = 1, \cdots, M, \\ \underline{u}_{it} - \mathscr{L}_i \underline{u}_i = f_i(t, x, \underline{u}_i; \underline{v}_i; \overline{w}_i), & i = M+1, \cdots, N, \end{cases} \quad (t, x) \in D_T, \tag{2.31}$$

$$\frac{\partial \underline{u}_i}{\partial \nu} = G_i(t, x, \underline{u}_i; \underline{V}_i; \overline{W}_i), \quad i = M+1, \cdots, N, \quad (t, x) \in \Gamma_T \tag{2.32}$$

$$\underline{u}_i(0, x) = \varphi_i(x), \qquad i = 1, \cdots, N, \quad x \in \Omega, \tag{2.33}$$

respectively. The proof in the case $M > 0$, $N \geq 2$ has not essential difficults and we omit it here.

In next section, our main objective is to prove that $(\overline{u}_1, \overline{u}_2, \cdots, \overline{u}_N) = (\underline{u}_1, \underline{u}_2, \cdots, \underline{u}_N)$ and uniqueness of the solution of the problem (2.1)—(2.3).

§ 3 Uniqueness of Solution

In this section, we shall prove the uniqueness of the solutions of the problem(1.1)—(1.3). For this, we consider the following problem:

$$\begin{cases} W_{it} = f_i^*(t, x, W_1, W_2, \cdots, W_{2N}), & i=1,\cdots,M;\ N+1,\cdots,N+M, \\ W_{it} - \mathscr{L}_i^* W_i = f_i(t, x, W_1, W_2, \cdots, W_{2N}), & i=M+1,\cdots,N;\ N+M+1,\cdots,2N, \end{cases} \quad (t,x)\in D_T, \tag{3.1}$$

$$\frac{\partial W_i}{\partial \nu} = G_i^*(t, x, W_1, W_2, \cdots, W_{2N}),\quad i=M+1,\cdots,N;\ N+M+1,\cdots,2N, (t,x)\in\Gamma_T, \tag{3.2}$$

$$W_i(0, x) = \varphi_i^*(x), \qquad i=1,\cdots,N,\ \ N+1,\cdots,2N,\ \ x\in\Omega, \tag{3.3}$$

where $W=(W_1, W_2, \cdots, W_{2N}) = (\overline{u_1}, \cdots, \overline{u_N}, \underline{u}_1, \cdots, \underline{u}_N) \in S(D_T)\times S(D_T)$;

$$f_i^*(t, x, W_1, W_2, \cdots, W_{2N}) = \begin{cases} f_i(t, x, \overline{u_i}; \overline{v_i}; \underline{w}_i), & i=1,\cdots,M,\ \ M+1,\cdots,N, \\ f_i(t, x, \underline{u}_i; \underline{v}_i; \overline{w_i}), & i=N+1,\cdots,N+M, N+M+1,\cdots,2N \end{cases} \tag{3.4}$$

$$G_i^*(t,x,W_1,W_2,\cdots,W_{2N}) = \begin{cases} G_i(t,x,\overline{u_i}; \overline{V_i}; \underline{W}_i), & i=1,\cdots,M,\ \ M+1,\cdots,N, \\ \hat{G}_i(t,x,\underline{u}_i; \underline{V}_i; \overline{W}), & i=N+1,\cdots,N+M, N+M+1,\cdots,2N, \end{cases} \tag{3.5}$$

$$\varphi_i^*(x) = \varphi_{N+i}^*(x) = \varphi_i(x),\ i=1,\cdots,N;\ \mathscr{L}_i^* = \mathscr{L}_{N+i}^* = \mathscr{L}_i\ \ i=M+1,\cdots,N. \tag{3.6}$$

Let $W=(W_1, W_2, \cdots, W_{2N})$ and $\widetilde{W}=(\widetilde{W}_1, \widetilde{W}_2, \cdots, \widetilde{W}_{2N})$ be two solutions of the problem (2.1)—(2.3). Set $V_i = W_i - \widetilde{W}_i$ $(i=1,\cdots,2N)$. Multiply both side of ith equation by V_i and integrat it over $D_t = \Omega\times(0, t)$ for any $0<t\le T$ to obtain

$$\sum_{i=1}^{M}\int_{D_t}\left[V_i\frac{\partial V_i}{\partial t} + V_{N+i}\frac{\partial V_{N+i}}{\partial t}\right]dx\,d\tau$$

$$=\sum_{i=1}^{M}\int_{D_t}\{V_i[f_i^*(\tau,x,W) - f_i^*(\tau,x,\widetilde{W})] + V_{N+i}[f_i^*(\tau,x,W) - f_i^*(\tau,x,\widetilde{W})]\}dx\,d\tau, \tag{3.7}$$

$$\sum_{i=M+1}^{N}\int_{D_t}\left[V_i\frac{\partial V_i}{\partial t} + V_{N+i}\frac{\partial V_{N+i}}{\partial t}\right]dx\,d\tau - \sum_{i=M+1}^{N}\int_{D_t}(V_i\,\mathscr{L}_i V_i + V_{N+i}\,\mathscr{L}_i V_{N+i})\,dx\,d\tau$$

$$=\sum_{i=M+1}^{N}\int_{D_t}\{V_i[f_i^*(\tau,x,W) - f_i^*(\tau,x,\widetilde{W})] + V_{N+i}[f_i^*(\tau,x,W) - f_i^*(\tau,x,\widetilde{W})]\}dx\,d\tau. \tag{3.8}$$

From (3.7), $V_i(0, x) = 0$ $(i=1,\cdots,2N)$ and the Lipschitz condition (2.10)、(2.11), we get

$$\frac{1}{2}\sum_{i=1}^{M}\int_{\Omega}[V_i^2(t,x) + V_{N+i}^2(t,x)]dx \le \sum_{i=1}^{M} c_i\int_{D_t}[\,|V_i|\,|W-\widetilde{W}| + |V_{N+i}|\,|W-\widetilde{W}|\,]dx\,d\tau$$

$$\le \sum_{i=1}^{M}\frac{1}{2}\int_{D_t}[V_i^2 + V_{N+i}^2]dx\,d\tau + \sum_{i=1}^{M}\frac{c_i^2}{2}\int_{D_t}|W-\widetilde{W}|^2 dx\,d\tau$$

$$\le \sum_{i=1}^{M}\left(\frac{1}{2} + \frac{1}{2}\sum_{j=1}^{M} c_j^2\right)\int_{D_t}(V_i^2 + V_{N+i}^2)\,dx\,d\tau + \sum_{i=M+1}^{N}\int_{D_t}(V_i^2 + V_{N+i}^2)\,dx\,d\tau. \tag{3.9}$$

Using the divergence theorem and condition (2.5), we have

$$\frac{1}{2}\sum_{i=M+1}^{N}\int_{\Omega}[V_i^2(t,x) + V_{N+i}^2(t,x)]dx + \sum_{i=M+1}^{N} a_i\int_{D_t}[\,|\nabla V_i|^2 + |\nabla V_{N+i}|^2]dx\,d\tau$$

$$\le \sum_{i=M+1}^{N}\int_0^t\int_{\partial\Omega}\left[V_i\frac{\partial V_i}{\partial \nu} + V_{N+i}\frac{\partial V_{N+i}}{\partial \nu}\right]ds\,d\tau + \sum_{i=M+1}^{N}\int_{D_t}\left[V_i\sum_{k=1}^{N} a_k^{(i)}\frac{\partial V_i}{\partial x_k} + V_{N+i}\sum_{k=1}^{n} a_k^{(i)}\frac{\partial V_{N+i}}{\partial x^k}\right]dx\,d\tau$$

$$+\sum_{i=M+1}^{N}\int_{D_t}\{V_i(f_i^*(\tau,x,W) - f_i^*(\tau,x,\widetilde{W})] + V_{N+i}[f_i^*(\tau,x,W) - f_i^*(\tau,x,\widetilde{W})]\}dx\,d\tau$$

$$\triangleq I_1 + I_2 + I_3. \tag{3.10}$$

We estimate each term of the right side of (3.10). The boundary conditions show

$$|I_1| = \Big|\sum_{i=M+1}^{N}\int_0^t\int_{\partial\Omega}[V_i[g_i(\tau,s,W) - g_i(\tau,s,W)] + V_{N+i}[g_i(\tau,s,W) - g_i(\tau,s,W)\}\mathrm{d}x\mathrm{d}\tau\Big|$$

$$\leq \sum_{i=M+1}^{N} C_i \int_0^t \int_{\partial\Omega}[V_i^2 + V_{N+i}^2]\mathrm{d}s\mathrm{d}\tau.$$

Applying the inverse imbedding theorem (see [5]) for any $u_i(x) \in H^2(\Omega)$,

$$\int_{\partial\Omega} u_i^2 \mathrm{d}s \leq \varepsilon_i \int_\Omega |\nabla u_i|^2 \mathrm{d}x + \frac{K_i}{\varepsilon_i}\int_\Omega |u_i|^2 \mathrm{d}x,$$

Where $\varepsilon_i > 0$, $K_i > 0$ is a constant depending only on Ω and $\partial\Omega$, we obtain

$$|I_1| \leq \sum_{i=M+1}^{N} C_i\{\varepsilon_i \int_{D_t}[|\nabla V_i|^2 + |\nabla V_{N+i}|^2]\mathrm{d}x\mathrm{d}\tau + \frac{K_i}{\varepsilon_i}\int_{D_t}[V_i^2 + V_{N+i}^2]\mathrm{d}x\mathrm{d}\tau\},$$

Choose $\varepsilon_i = \dfrac{a_i}{4C_i}$, hen

$$|I_1| \leq \sum_{i=M+1}^{N}\Big\{\frac{a_i}{4}\int_{D_t}[|\nabla V_i|^2 + |\nabla V_{N+i}|^2]\mathrm{d}x\mathrm{d}\tau + \frac{4C_i^2 K_i}{a_i}\int_{D_t}[V_i^2 + V_{N+i}^2]\mathrm{d}x\mathrm{d}\tau\Big\}. \tag{3.11}$$

Next, we estimate I_2,

$$|I_2| \leq \sum_{i=M+1}^{N}\int_{D_t}\Big[|V_i|\Big(\sum_{k=1}^{n}|a_k^{(i)}|^2\Big)^{1/2}|\nabla V_i| + |V_{N+i}|\Big(\sum_{k=1}^{n}|a_k^{(i)}|^2\Big)^{1/2}|\nabla V_{N+i}|\Big]\mathrm{d}x\mathrm{d}\tau \tag{3.12}$$

$$\leq \sum_{i=M+1}^{N}\frac{a_i}{4}\int_{D_t}[|\nabla V_i|^2 + |\nabla V_{N+i}|^2]\mathrm{d}x\mathrm{d}\tau + \sum_{i=M+1}^{N}\frac{1}{a_i}\max\Big(\sum_{k=1}^{n}|a_k^{(i)}|^2\Big)^{1/2}\int_{D_t}(V_i^2 + V_{N+i}^2)\mathrm{d}x\mathrm{d}\tau.$$

Finally, as the argument in the proof of the equality (3.9), it's clear that

$$|I_3| \leq \sum_{i=M+1}^{N}\Big(\frac{1}{2} + \sum_{i=M+1}^{N}\frac{C_j^2}{2}\Big)\int_{D_t}(V_i^2 + V_{N+i}^2)\mathrm{d}x\mathrm{d}\tau + \Big(\frac{1}{2}\sum_{j=M+1}C_j^2\Big)\sum_{i=1}^{M}[V_i^2 + V_{N+i}^2]\mathrm{d}x\mathrm{d}\tau. \tag{3.13}$$

Instead of the right side of (3.10) by the estimates (3.11), (3.12) and (3.13), and combine with the same terms, we get

$$\frac{1}{2}\sum_{i=M+1}^{N}\int_\Omega[V_i^2(t,x) + V_{N+i}^2(t,x)]\mathrm{d}x + \frac{1}{2}\sum_{i=M+1}^{N} a_i\int_{D_t}[|\nabla V_i|^2 + |\nabla V_{N+i}|^2]\mathrm{d}x\mathrm{d}\tau$$

$$\leq \sum_{i=M+1}^{N}\Big[\frac{4C_i^2K_i}{a_i} + \frac{1}{a_i}\max\Big(\sum_{k=1}^{n}|a_k^{(i)}|^2\Big)^{1/2} + \Big(\frac{1}{2} + \sum_{i=M+1}^{N}\frac{C_i^2}{2}\Big)\Big]\int_{D_t}[V_i^2 + V_{N+i}^2]\mathrm{d}x\mathrm{d}\tau$$

$$+\Big(\frac{1}{2}\sum_{j=M+1}C_j^2\Big)\sum_{i=1}^{M}\int_{D_t}[V_i^2 + V_{N+i}^2]\mathrm{d}x\mathrm{d}\tau. \tag{3.14}$$

Adding above inequality to the inequality (3.9), we have the estimate

$$\frac{1}{2}\sum_{i=1}^{2N}\int_\Omega V_i^2(t,x)\mathrm{d}x + \frac{1}{2}\sum_{i=M+1}^{N} a_i\int_{D_t}[|\nabla V_i|^2 + |\nabla V_{N+i}|^2]\mathrm{d}x\mathrm{d}\tau$$

$$\leq \sum_{i=1}^{M}\Big(\frac{3}{2} + \sum_{j=1}^{M}C_j^2\Big)\int_{D_t}(V_i^2 + V_{N+i}^2)\mathrm{d}x\mathrm{d}\tau$$

$$+\sum_{i=M+1}^{N}\frac{1}{2}\Big[\frac{8C_i^2K_i}{a_i}+\frac{2}{a_i}\max\big(\sum_{k=1}^{n}|a_k^{(i)}|^2\big)^{1/2}+\big(1+\sum_{j=M+1}^{N}C_j^2\big)\Big]\int_{D_t}[V_i^2+V_{N+i}^2]\,\mathrm{d}x\mathrm{d}\tau.$$

Dropping the second term in the left side, we get

$$\sum_{i=1}^{2N}\int_\Omega V_i^2(t,x)\,\mathrm{d}x\le\sigma\sum_{i=1}^{2N}\int_0^t\int_\Omega V_i^2(t,x)\,\mathrm{d}x\mathrm{d}\tau,$$

here

$$\sigma=\max\Big\{(3+\sum_{j=1}^{M}C_j^2),\ \frac{8C_i^2K_i}{a_i}+\frac{2}{a_i}\max\big(\sum_{k=1}^{n}|a_k^{(i)}|^2\big)^{1/2}+\big(1+\sum_{i=M+1}^{N}C_j^2\big),\ i=M+1,\cdots,N\Big\}.$$

The Grownwall's inequality implies

$$e^{-\sigma t}\Big(\sum_{i=1}^{2N}\int_\Omega V_i^2(t,x)\,\mathrm{d}x\Big)\le 0,\quad\forall\ 0\le t\le T.$$

Therefore $V_i(t,x)=0$, i.e., $W_i=\widetilde{W}_i$, $i=1,\cdots,2N$. This proves that the problem (3.1)—(3.3) has at least one solution.

Suppose $W=(W_1,W_2,\cdots,W_{2N})$ is a solution of (3.1)—(3.3), The function $U=(\overline{u}_1,\cdots,\overline{u}_N,\underline{u}_1,\cdots,\underline{u}_N)$ is also a solution of (3.1)—(3.3). By uniqueness, $U=W$. On the other hands by the same argument as in the proof of (3.1)—(3.3), the problem (2.1)—(2.3) has at least one solution $u=(u_1,u_2,\cdots,u_N)$. Clearly $U=(u_1,\cdots,u_N,u_1,\cdots,u_N)$ is also a solution of problem (3.1)—(3.3) and therefore $U=W=U$. This lead, $\overline{u}_i=\underline{u}_i=u_i$. Hence $(\overline{u}_1,\cdots,\overline{u}_N)=(\underline{u}_1,\cdots,\underline{u}_N)$ is an unique solution of the original problem (2.1)—(2.3). This completed the proof of Theorem.

§ 4 Large Time Behavior of Solution

In this section, we consider large time behavior of the solution of the nonlinear reaction diffusion equation

$$\frac{\partial u}{\partial t}=\frac{1}{\varepsilon^2}\nabla(D(x,t)\nabla u)+\frac{1}{\varepsilon}\sum_{j=1}^{n}A_j(x,t)\frac{\partial u}{\partial x_j}+f(x,t,u),\quad(x,t)\in Q=\Omega\times(0,\infty),\tag{4.1}$$

$$\frac{\partial u}{\partial n}\Big|_{\Gamma_1}=0,\qquad\Big(P\frac{\partial u}{\partial n}+Qu\Big)\Big|_{\Gamma_2}=0,\qquad\Gamma_i=\partial\Omega_i\times(0,\infty),\ i=1,2,\tag{4.2}$$

$$u(x,t)\big|_{t=0}=\varphi(x),\qquad x\in\Omega,\tag{4.3}$$

where $\partial\Omega_1\cup\partial\Omega_2=\partial\Omega$, $\mathrm{mes}\,\partial\Omega_2>0$, $D(x,t)$, $A_j(x,t)$, $P=P(x,t)$, $Q=Q(x,t)$ are continuous $N\times N$ matrix-valued functions on $(x,t)\in\overline{\Omega}\times[0,\infty)$ and $D(x,t)$ is a positive definite matrix. We denote $0<d\le d(x,t)$ the smallest eigenvalue of the matrix $D(x,t)$ and $M_i=\max\limits_{\Omega\times(0,\infty)}|A_i(x,t)|$. $u(x,t)=(u_1(x,t),\cdots,u_N(x,t))$, $f(x,t,u)=(f_1(x,t,u),\cdots,f_N(x,t,u))$ and $\varphi(x)=(\varphi_1(x),\cdots,\varphi_N(x))$ are N-dimensional vector-valued functions. Assum $f(x,t,u)$ satisfies the Lipschitz condition

$$|f_i(x,t,u_1,\cdots,u_N)-f_i(x,t,v_1,\cdots,v_N)|\le K_i\Big(\sum_{j=1}^{N}|u_j-v_j|^2\Big)^{1/2}\tag{4.4}$$

for all $(x,t)\in Q$ and every $u,v\in S$, where S is a bounded set in $\mathbf{R}^N$. Set

$$K=\left(\sum_{i=1}^{N} K_i^2\right)^{1/2}, \quad M=\left(\sum_{i=1}^{n} M_i^2\right)^{1/2}.$$

In ovder to prove the main result, we need

Lemma 4.1 Let $\Omega \subseteq \mathbf{R}^n$, $\partial\Omega=\partial\Omega_1 \cup \partial\Omega_2$, where mes $\partial\Omega_2>0$. Assume $u(x,t)\in H^2(\Omega)$ and

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma_1}=0, \qquad \left.\left(P\frac{\partial u}{\partial n}+Qu\right)\right|_{\Gamma_2}=0,$$

where $\frac{\partial}{\partial n}$ is outward directional derivative on $\partial\Omega$. If $\det Q \neq 0$, then there exists a constant $\mu>0$ depending only on Ω such that

$$\int_\Omega |\nabla u(x,t)|^2 dx \geq \mu \int_\Omega |u(x)|^2 dx$$

(see [8], Lemma 3.6.4).

Theorem 4.2 Let $\varepsilon>0$ be small such that

$$2\sigma = d\mu - \varepsilon^2\left(\frac{M^2}{d}+2K\right)>0. \tag{4.5}$$

Assume $f(x,t,0)=0$ and condition (4.4) be satisfied. If

$$DQ^{-1}P>0 \quad \text{on } \Gamma_2, \tag{4.6}$$

then

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq \|\varphi\|_{L^2(\Omega)} e^{-\frac{\sigma}{\varepsilon^2}t}. \tag{4.7}$$

Proof Let $u=(u_1,\cdots,u_N)$ be a solution of the problem (4.1)—(4.3). We sufficiently prove it for the smooth veetor function. Set

$$\Phi(t)=\frac{1}{2}\int_\Omega \langle u,u\rangle dx,$$

where $\langle u,v\rangle$ denote by the inner product of the vectors u,v. By (4.1) and (4.2) we have

$$\frac{d\Phi(t)}{dt}=\int_\Omega \langle u,u_t\rangle dx=\frac{-1}{\varepsilon^2}\int_\Omega \langle \nabla u, D\nabla u\rangle dx+\frac{1}{\varepsilon}\sum_{j=1}^{N}\int_\Omega \langle u, A_j\frac{\partial u}{\partial x_j}\rangle dx$$

$$+\int_\Omega \langle u, f(x,t,u)\rangle dx+\frac{1}{\varepsilon^2}\int_{\partial\Omega_2}\langle u, D\frac{\partial u}{\partial n}\rangle ds.$$

Since $d(x,t)\geq d>0$ and (4.4) we get

$$\frac{d\Phi(t)}{dt}\leq -\frac{d}{\varepsilon^2}\int_\Omega \langle \nabla u,\nabla u\rangle dx+\frac{1}{\varepsilon}\int_\Omega |u|\left(\sum_{j=1}^{n}|A_i|^2\right)^{1/2}|\nabla u|dx$$

$$+\int_\Omega |u|\,|f(x,t,u)-f(x,t,0)|dx-\frac{1}{\varepsilon^2}\int_{\partial\Omega_2}\langle \frac{\partial u}{\partial n}, DQ^{-1}P\frac{\partial u}{\partial n}\rangle ds$$

$$\leq -\frac{d}{\varepsilon^2}\int_\Omega |\nabla u|^2 dx+\frac{1}{\varepsilon}\left(\eta\int_\Omega |\nabla u|^2 dx+\frac{1}{4\eta}\left(\sum_{i=1}^{n}M_i^2\right)\int_\Omega |u|^2 dx\right)+K\int_\Omega |u|^2 dx.$$

Choose $\eta=\frac{d}{2\varepsilon}$, then

$$\frac{d\Phi(t)}{dt}\leq -\frac{d}{2\varepsilon^2}\int_\Omega |\nabla u|^2 dx+\left(\frac{M^2}{2d}+K\right)\int_\Omega |u|^2 dx.$$

Using Lemma 4.1, we obtain

$$\frac{d\Phi(t)}{dt} \le -\frac{1}{2\varepsilon^2}(d\mu - \varepsilon^2(\frac{M}{d}+2K))\int_\Omega |u|^2 dx = -\frac{\sigma}{\varepsilon^2}\Phi(t).$$

Hence (4.7) hold.

(4.7) shows that the solution of the problem (4.1)—(4.3) are in asymptotic state.

We now consider the Neumann boupdary value problem

$$\frac{\partial u_i}{\partial t} = \nabla(D_i(x,t)\nabla u_i) + f_i(x,t,u), \quad (x,t) \in Q = \Omega \times (0,\infty), \tag{4.8}$$

$$\frac{\partial u_i}{\partial n} = 0, \quad (x,t) \in \Gamma = \partial\Omega \times (0,\infty), \tag{4.9}$$

$$u_i(x,t)\big|_{t=0} = \varphi_i(x), \quad x \in \Omega,\ i = 1,\cdots,N. \tag{4.10}$$

Suppose $D_i(x,t) \ge d > 0$ are continuous on Ω, $f_i(x,t,u)(i=1,\cdots,N)$ satisfy condition (4.4).

Lemma 4.3 Suppose λ is the smallest positive eigen value of the problem

$$\begin{cases} \nabla(D\nabla u) = 0, \\ \frac{\partial u}{\partial n}\Big|_\Gamma = 0. \end{cases}$$

then

$$\|\nabla u\|_{L^2(\Omega)} \ge \lambda \|u - \bar{u}\|_{L^2(\Omega)},$$

Where $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u(x)dx$ (see the Appendix in [6]).

Theorem 4.4 Suppose that $f_i(x,t,u) \le 0$ for all $(x,t) \in \bar{\Omega} \times [0,\infty)$ $u = (u_1,\cdots,u_N) \in S \subseteq \mathbf{R}^N$. Aussum $f_i(x,t,u(x,t)) \in L^2(Q)$. If the problem (4.8)—(4.10) has a solution $u(x,t) \in H^2(Q) = L^2((0,\infty), H^2(\Omega))$ for $\varphi(x) \in L^2(\Omega)$. Then there exist constants

$$c_i = \frac{1}{|\Omega|}\int_\Omega \varphi_i(x)dx + \frac{1}{|\Omega|}\int_0^\infty\int_\Omega f_i(x,\tau,u(x,\tau))dxd\tau \tag{4.11}$$

such that

$$\|u_i(\cdot,t) - c_i\|_{L^2(\Omega)} \to 0 \text{ as } t \to \infty,\ i = 1,\cdots,N. \tag{4.12}$$

Proof Denote spatial average of the function $u_i(x,t)$ and $\varphi_i(x)$ by

$$\overline{u_i}(t) = \frac{1}{|\Omega|}\int_\Omega u_i(x,t)dt, \quad \overline{\varphi_i} = \frac{1}{|\Omega|}\int_\Omega \varphi_i(x)dx, \quad i = 1,\cdots,N, \tag{4.13}$$

respectively. Integrating over Ω, applying the divegence theorem and using the boundary condition (4.9), we abtain

$$\frac{d\overline{u_i}(t)}{dt} = \overline{f_i}(t) = \frac{1}{|\Omega|}\int_\Omega f_i(x,t,u(x,t))dx, \quad i = 1,\cdots,N. \tag{4.14}$$

Since $f_i \le 0$, so that $\overline{f_i}(t) \le 0$ and $\overline{u_i}(t)$ is a nondecreasing function in t. Therefore there are N constants $c_1,\cdots,c_N$ such that $\lim_{t\to\infty}\overline{u_i}(t) = c_i$ $(i=1,\cdots,N)$. From this and the equality

$$\overline{u_i}(t)=\overline{\varphi_i}+\frac{1}{|\Omega|}\int_0^t\int_\Omega f_i(x,\tau u(x,\tau)\,\mathrm{d}x\mathrm{d}\tau$$

ensures that the integral

$$\int_0^\infty\int_\Omega f_i(x,\tau,u(x,\tau))\,\mathrm{d}x\mathrm{d}\tau$$

exists, and that

$$c_i=\overline{\varphi_i}+\frac{1}{|\Omega|}\int_0^\infty\int_\Omega f_i(x,t,u(x,\tau))\,\mathrm{d}x\mathrm{d}\tau;\quad i=1,\cdots,N.$$

We set

$$\Phi(t)=\frac{1}{2}\int_\Omega\langle u-\overline{u},u-\overline{u}\rangle\mathrm{d}x,$$

then by (4.8)、(4.9) and divergence theorem we get

$$\begin{aligned}\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t}&=\int_\Omega\langle u-\overline{u},u_t-\overline{u_t}\rangle\mathrm{d}x\\&=\sum_{i=1}^N\int_\Omega\langle u_i-\overline{u_i},\nabla(D_i\nabla(u-\overline{u}))\rangle\mathrm{d}x+\int_\Omega\langle u-\overline{u},f-\overline{f}\rangle\mathrm{d}x\\&\le-\sum_{i=1}^N\int_\Omega\langle\nabla(u_i-\overline{u_i}),D_i\nabla(u_i-\overline{u_i})\rangle\mathrm{d}x+\int_\Omega|u-\overline{u}||f-\overline{f}|\mathrm{d}x\\&\le-d\int_\Omega|\nabla(u-\overline{u})|^2\mathrm{d}x+\frac{\mathrm{d}\lambda}{2}\|u-\overline{u}\|^2_{L^2(\Omega)}+\frac{1}{2d\lambda}\int_\Omega|f-\overline{f}|^2\mathrm{d}x.\end{aligned}$$

Lemma 4.3 implies

$$\begin{aligned}\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t}&\le-\frac{d\lambda}{2}\|u-\overline{u}\|^2_{L^2(\Omega)}+\frac{1}{2d\lambda}\int_\Omega|f-\overline{f}|^2\mathrm{d}x\\&=-d\lambda\Phi(t)+\frac{1}{2d\lambda}\|f-\overline{f}\|^2_{L^2(\Omega)}.\end{aligned}$$

Integrating this differential inequality, we obtain

$$\|u(\cdot,t)-\overline{u}(t)\|^2_{L^2(\Omega)}\le e^{-d\lambda t}\|\varphi(\cdot)-\overline{\varphi}\|^2_{L^2(\Omega)}+\frac{1}{d\lambda}\int_0^t\|f-\overline{f}\|^2_{L^2(\Omega)}e^{-d\lambda(t-\tau)}\mathrm{d}\tau.\tag{4.15}$$

We will prove that

$$\int_0^t\|f-\overline{f}\|^2_{L^2(\Omega)}e^{-d\lambda(t-\tau)}\mathrm{d}\tau\to0\ \text{as}\ t\to+\infty.\tag{4.16}$$

In fact, giving $\varepsilon>0$, then by $f_i\in L^2(Q)$ there exists T large enough such that

$$\int_T^\infty\|f-\overline{f}\|^2_{L^2(\Omega)}\mathrm{d}\tau<\frac{\varepsilon}{2}$$

and

$$\int_0^T\|f-\overline{f}\|^2_{L^2(\Omega)}e^{-d\lambda(t-\tau)}\mathrm{d}\tau\le e^{-\lambda dT}\int_0^T\|f-\overline{f}\|^2_{L^2(\Omega)}\mathrm{d}\tau<\frac{\varepsilon}{2}\ \text{ when }t>2T.$$

Thus

$$\int_0^t\|f-\overline{f}\|^2_{L^2(\Omega)}e^{-d\lambda(t-\tau)}\mathrm{d}\tau\le\int_0^T\|f-\overline{f}\|^2_{L^2(\Omega)}e^{-d\lambda(t-\tau)}\mathrm{d}\tau+\int_T^\infty\|f-\overline{f}\|e^{-d\lambda(t-\tau)}\mathrm{d}\tau<\varepsilon$$

when $t>2T$. Here we have used the following estimate

$$\int_0^t\|f-\overline{f}\|^2_{L^2(\Omega)}e^{-d\lambda(t-\tau)}\mathrm{d}\tau\le\int_0^\infty\|f-\overline{f}\|^2_{L^2(\Omega)}\mathrm{d}\tau\le2\int_0^\infty\|f\|^2_{L^2(\Omega)}\mathrm{d}\tau$$

$$+2\int_0^{\infty}\|\overline{f}\|^2_{L^2(\Omega)}d\tau \le 4\int_0^{\infty}\|f\|^2_{L^2(\Omega)}d\tau < \infty.$$

From (4.15) and (4.16) we have

$$\lim_{t\to\infty}\|u(\cdot,t)-\overline{u}(t)\|_{L^2(\Omega)}=0. \tag{4.17}$$

so that

$$\sum_{i=1}^{N}\|u_i(\cdot,t)-c_i\|_{L^2(\Omega)} \le \sum_{i=1}^{N}\|u_i(\cdot,t)-\overline{u_i}(t)\|_{L^2(\Omega)}+\sum_{i=1}^{N}\|\overline{u_i}(t)-c_i\|_{L^2(\Omega)}$$

$$=\|u(\cdot,t)-\overline{u}(t)\|_{L^2(\Omega)}+\sum_{i=1}^{N}|\Omega||\overline{u}_i(t)-c_i|\to 0 \text{ as } t\to+\infty.$$

The proof of Theorem 4.4 is completed.

References

[1] Franz Rothe, Global Solutions of Reaction Diffusion Systems, Springer-Verlag, Berlin (1984).

[2] C. V. Pao, J. Diff. Equs., Vol.22, No.1, 145—163 (1976).

[3] C. V. Pao, Nonlinear Analysis, T. M. A., Vol.5, No.10, 1077—1094 (1981).

[4] Paul, D. Norman, Nonlinear Analysis T. M. A., Vol.4, 203—206 (1980).

[5] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice Hall Englewood Cliffs N. J. (1964).

[6] E. Conway, D. Hopf & J. Smoller, SIAM, J. Appl. Math., 35, 1—16 (1978).

[7] Su, Yu, Nonlinear Analysis, T. M. A., Vol.9, No.3, 275—288 (1985).

[8] Charles B. Morrey, Multiple Integrals in the Calculus of Variations, Springer-Verlag, New York (1966).