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Large Deviations for the Boundary Crossing Probabilities of Some Random Fields

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Abstract

For $Y_1, Y_2, \cdots$ i.i.d. with $Y_1 \sim N(\mu, 1)$ and $S_n = \sum_1^n Y_i$, the large deviations are obtained for the probabilities that $\max_{m_0 \le l < k \le m_1} (S_k - S_l)/[(k-l)(\frac{k-l}{m})]^{1/2} \geqslant b$ conditionally given (i) $S_m = 0$, and (ii) $S_{m_1} = \xi$. Applied these results to the double change-points model with some nuisance parameters, we developed the large deviation for the significance level of the likelihood ratio test.

I. Introduction

Let $X_1, \cdots, X_m$ be independent random variable, and $X_i \sim N(\mu^{(i)}, 1)$, $1 \leqslant i \leqslant m$;

(1) $H_0: \mu^{(1)} = \cdots = \mu^{(m)} = \mu_0$

$H_1: 1 \leqslant \rho_1 < \rho_2 \leqslant m$ such that $\mu^{(1)} = \cdots = \mu^{(\rho_1)} = \mu_0$;

$\mu^{(\rho_1+1)} = \cdots = \mu^{(\rho_2)} = \mu_0 + \delta$, $\mu^{(\rho_2+1)} = \cdots = \mu^{(m)} = \mu_0$.

In above hypotheses on double change-points (ρ_1, ρ_2), if μ_0, δ are given, the log likelihood ratio test statistic is

(2) $$\max_{1 \le l < k \le m} \delta[\widetilde{S}_j - j(\mu_0 + \frac{\delta}{2}) - (\widetilde{S}_i - i(\mu_0 + \frac{\delta}{2}))]$$

where $\widetilde{S}_k = \sum_{i=1}^k X_i$. Define

(3) $$T_1 = \inf\{k : \max_{1 \le l < k} \delta[\widetilde{S}_k - k(\mu_0 + \frac{\delta}{2}) - (\widetilde{S}_l - l(\mu_0 + \frac{\delta}{2}))] \geqslant b\}$$

then the significance level is

$$P(T_1 \leqslant m | H_0)$$

which is the probability that a two dimensional Gaussian random field crosses the constant boundary b. Hogan and Siegmund [1] adopted the method by Pickands [2] etc. to obtain explicit large deviation for this boundary crossing probability. Siegmund [3] developed woodroofe's [4] method to preset a similar result for one-parameter exponential family.

Typically μ_0 and δ are unknown nuisance parameters in most applications. To avoid some mathematical difficulties, usually substitute $(\hat{\mu}_0, \delta_0)$ for (μ_0, δ) in (2), where $\hat{\mu}_0 = \widetilde{S}_m/m$ is the maximum likelihood estimator for μ_0 under H_0 and δ_0 is a

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threshold value of δ, which one is interested in detecting (cf. Siegmund[5], 3.6). Then (3) becomes

$$T_2=\inf\{k:\max_{1\le l<k}\delta_0[\tilde{S}_k-k\tilde{S}_m/m-(\tilde{S}_l-l\tilde{S}_m/m)-(k-l)\delta_0/2]\geq b\}.$$

$P(T_2\leq m|H_0)$ is the conditional probability that a random field crosses the boundary b. Hogan and Siegmund [1], Siegmund [3] developed its large deviation approximation.

In this paper, we try to study the likelihood ratio test for hypotheses (1) with both unknown μ_0 and δ. In this case, the likelihood ratio statistic is

$$(4)\qquad \max_{1\le l<k\le m}|\tilde{S}_k-\tilde{S}_l-\frac{k-l}{m}\tilde{S}_m|/\sqrt{(k-l)(1-\frac{k-l}{m})}$$

To get the significance level leads to develop a conditional probability that a two dimensional Gaussian field crosses the non-linear boundary $b\sqrt{(k-l)(1-\frac{k-l}{m})}$

It seems to me that this topic has not been treated in the literature before. We try to solve this problem in this paper.

Let P_μ denote the probability measure which makes $Y_1, Y_2,\cdots$ i.i.d. with $Y_1\sim N(\mu,1)$, $S_0=0, S_n=\sum_1^n Y_i$, and $P_\xi^{(n)}(\cdot)=P_0(\cdot|S_n=\xi)$. For $1<m_0<m_1<m$, $b>0$, define

$$(5)\qquad \tau=\inf\{k\geq m_0:\max_{1\le l<k-m_0}(S_k-S_l)/\sqrt{(k-l)(1-\frac{k-l}{m})}\geq b\}$$

$$(6)\qquad T=\inf\{k\geq m_0:\max_{1\le l<k-m_0}|S_k-S_l|/\sqrt{(k-l)(1-\frac{k-l}{m})}\geq b\}$$

Although the statistical inference for (ρ_1,ρ_2) is only relevent to T, τ is more tractable than T. Moreover, it is easy to get the similar version for T from some results for τ. The main results of this paper is stated in Section 2. Theorem 1 and Theorem 2 present the large deviations for $P_0^{(m)}(\tau\leq m_1)$ and $P_\xi^{(m_1)}(\tau<m_1)$ respectively. The complicated proofs of these two theorems are delayed to Section 3 and Section 4. Corollary 1 and Corollary 2 are alike somewhat in form to the related Theorem 11.30 of Siegmund [6] and Theorem 3.11 of Siegmund [5], that are proved by a method which dose not seem to suitable to random fields. The method we adapt was presented originally by Woodroofe [4] and developed to random field by Siegmund [3]. Applied these results, we discuss the likelihood ratio test for hypotheses (1) and get the large deviation for the significance level.

I discovered after writing this paper that the revised version of [3] provided the very similar result to Theorem 1. The differency in forms only comes from the slightly different sets over which we maximize. I am grateful to Prof. David

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2. Main Results

Theorem 1 Assume that $b=\mu_1 m^{1/2}$, $m_0=t_0 m$, $m_1=t_1 m$ with $\mu_1>0$, $0<t_0<t_1<1$. Then as $m\to\infty$,

$$(7)\quad P_0^{(m)}(\tau\leqslant m_1)\sim\frac{m}{2}b\varphi(b)\cdot\int_{\mu_1(t_1^{-1}-1)^{1/2}}^{\mu_1(t_0^{-1}-1)^{1/2}}\frac{1}{x}(t_1-\mu_1^2\frac{1-t_1}{x^2})(x^2+\mu_1^2)[\nu(x+\frac{\mu_1^2}{x})]^2dx$$

where $\varphi(x)=(2\pi)^{-(1/2)}\exp(-x^2/2),\Phi(x)=\int_{-\infty}^{x}\varphi(u)du$,

$$(8)\quad \nu(x)=2x^{-2}\exp\{-2\sum_{1}^{\infty}n^{-1}\Phi(-\frac{x}{2}n^{1/2})\}.$$

Corollary 1 Under the assumptions of Theorem 1,

$$(9)\quad P_0^{(m)}(T\leqslant m_1)\sim mb\varphi(b)\cdot\int_{\mu_1(t_1^{-1}-1)^{1/2}}^{\mu_1(t_0^{-1}-1)^{1/2}}\frac{1}{x}(t_1-\mu_1^2\frac{1-t_1}{x^2})(x^2+\mu_1^2)[\nu(x+\frac{\mu_1^2}{x})]^2dx.$$

The proofs of Theorem 1 and Corollary 1 will be presented in Section 3.

Theorem 2 Assume that $b=\mu_1 m^{1/2}$, $m_0=t_0 m$, $m_1=t_1 m$, $\xi=\xi_0 m$ with $\mu_1>0$, $0<t_0<t_1<1$, $\xi_0\in(\mu_1(1-t_1)\sqrt{\frac{t_0}{1-t_0}}, \mu_1\sqrt{t_1(1-t_1)})$. Then as $m\to\infty$,

$$(10)\quad P_\xi^{(m_1)}(\tau<m_1)\sim\frac{m}{2}\exp\{-\frac{m}{2}[\mu_1^2-\frac{\xi_0^2}{t_1(1-t_1)}]\}\sqrt{t_1(1-t_1)}$$

$$\times\frac{\mu_1}{\xi_0}[\frac{t_1(1-t_1)}{\xi_0}\mu_1^2-\xi_0](\frac{1-t_1}{\xi_0}\mu_1^2+\frac{\xi_0}{1-t_1})[\nu(\frac{1-t_1}{\xi_0}\mu_1^2+\frac{\xi_0}{1-t_1})]^2$$

where $\nu(\cdot)$ is given by (7).

Corollary 2 Assume that $b=\mu_1 m^{1/2}$, $m_0=t_0 m$, $m_1=t_1 m$, $\xi=\xi_0 m$ with $\mu_1>0$, $0<t_0<t_1<1$ $|\xi_0|\in(\mu_1(1-t_1)\sqrt{\frac{t_0}{1-t_0}}, \mu_1\sqrt{t_1(1-t_1)})$. Then as $m\to\infty$.

$$(11)\quad P_\xi^{(m_1)}(T<m_1)\sim\frac{m}{2}\exp\{-\frac{m}{2}[\mu_1^2-\frac{\xi_0^2}{t_1(1-t_1)}]\}\sqrt{t_1(1-t_1)}.$$

$$\times\frac{\mu_1}{|\xi_0|}[\frac{t_1(1-t_1)}{|\xi_0|}\mu_1^2-|\xi_0|](\frac{1-t_1}{|\xi_0|}\mu_1^2+\frac{|\xi_0|}{1-t_1})[\nu(\frac{1-t_1}{|\xi_0|}\mu_1^2+\frac{|\xi_0|}{1-t_1})]^2$$

The proofs of Theorem 2 and Corollary 2 will be presented in Section 4.

Now we want to discuss the likelihood ratio test for hypotheses (1) with unknown μ_0 and δ.

As only limiting behavior will be discussed, we assume that ρ_1, $\rho_2-\rho_1$, $m-\rho_2$ are effectively infinitely large. On the other hand, it is intrinsically difficult to detect (ρ_1,ρ_2) when ρ_1 occurs near 1, ρ_2 occurs near m, or $\rho_2-\rho_1$ sufficiently small (cf. Siegmund [5], 3.4).

From (4) and above assumptions, the likelihood ratio test statistic for hypotheses (1) may be taken as follows

(12) $$\max_{\substack{1\le l<k\le m_1\\ k-l\ge m_0}} |\widetilde{S}_k-\widetilde{S}_l-\frac{k-l}{m}\widetilde{S}_m|/\sqrt{(k-l)(1-\frac{k-l}{m})}$$

for some $1<m_0<m_1<m$. Hence the significance level is

(13) $$P\{\max_{\substack{1\le l<k\le m_1\\ k-l\ge m_0}} \frac{|\widetilde{S}_k-\widetilde{S}_l-\frac{k-l}{m}\widetilde{S}_m|}{\sqrt{(k-l)(1-\frac{k-l}{m})}} \geqslant b|H_0\}$$

where $b>0$ is a constant. Since $\widetilde{S}_k-\widetilde{S}_l$ is independent to $\widetilde{S}_m$ for all $1\leqslant l<k\leqslant m$, (12) equals

$$P_0^{(m)}\{\max_{\substack{1\le l<k\le m_1\\ k-l\ge m_0}} |S_k-S_l|/\sqrt{(k-l)(1-\frac{k-l}{m})}\geqslant b\} = P_0^{(m)}(T\leqslant m_1)$$

Thus Corollory 1 offers the large deviation for the significance level (12).

3. Proof of Theorem 1

In this Section, we always use the notation of Theorem 1. Moreover, $\mathscr{L}(x|c)$ indicates the distribution of random variable x under the condition c. The proof of Theorem 1 is given in a series of Lemmas.

Lemma 1 Assume $m_0\leqslant n\leqslant m_1$. Then as $m\to\infty$,

(i) uniformly in n and $x\leqslant(\log m)^{1/3}$

(14) $$P_0^{(m)}\{S_n\in b[n(1-\frac{n}{m})]^{1/2}+dx\}\sim[2\pi n(1-\frac{n}{m})]^{-\frac{1}{2}}e^{-\frac{1}{2}b^2}\exp\{-[\frac{n}{m}(1-\frac{n}{m})]^{-\frac{1}{2}}\mu_1 x\}dx$$

(ii) uniformly in n and $x>(\log m)^{1/3}$

(15) $$P_0^{(m)}\{S_n\geqslant b[n(1-\frac{n}{m})]^{\frac{1}{2}}+x\} = o(m^{-\frac{1}{2}}e^{-\frac{1}{2}b^2}).$$

Froof From $\mathscr{L}(S_n|P_0^{(m)}) = N(0,m(1-\frac{n}{m}))$,

(16) $$P_0^{(m)}\{S_n\in b[n(1-\frac{n}{m})]^{\frac{1}{2}}+dx\}$$

$$=[2\pi n(1-\frac{n}{m})]^{-\frac{1}{2}}e^{-\frac{1}{2}b^2}\exp\{-\frac{x^2}{2n(1-n/m)}-\frac{bx}{[n(1-n/m)]^{1/2}}\}dx$$

(14) and (15) follow (16) immediately.

Lemma 2 As $m\to\infty$, for $x\leqslant(\log m)^{1/3}$ uniformly

$$P_0\{S_j<b[j(1-\frac{j}{m})]^{1/2},\text{ for all } m_0<j\leqslant m_1-(\log m)^2|S_{m_1}=b[m_1(1-\frac{m_1}{m})]^{1/2}+x\}\to 1.$$

Proof $P_0\{S_j\geqslant b[j(1-\frac{j}{m})]^{1/2},\text{for some } m_0<j\leqslant m_1-(\log m)^2|S_{m_1}=b[m_1(1-\frac{m_1}{m})]^{1/2}+x\}$

$$=m\cdot\max_{m_0<j\le m_1-(\log m)^2}P_0\{S_j\geqslant b[j(1-\frac{j}{m})]^{1/2}|S_{m_1}=b[m_1(1-\frac{m_1}{m})]^{1/2}+x\}$$

$$=m\cdot\max_{m_0<j\le m_1-(\log m)^2}P_{j,m}.$$

Then

$$P_{j,m}=\int_{b[j(1-\frac{j}{m})]^{1/2}}^{\infty}[2\pi j(1-\frac{j}{m_1})]^{-\frac{1}{2}}\exp\{-\frac{1}{2j(1-j/m_1)}[y-\frac{j}{m_1}b[m_1(1-\frac{m_1}{m})]^{1/2}$$

$$-\frac{j}{m_1}x\}^2\}dy=\int_{c_{j,m}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt$$

where

$$c_{j,m}=\frac{b\{[j(1-j/m)]^{1/2}-j[m_1(1-m_1/m)]^{1/2}/m_1\}-jx/m_1}{[j(1-j/m_1)]^{1/2}}$$

$$\geqslant b(1-\frac{j}{m_1})^{1/2}/[(1-\frac{j}{m})^{1/2}+\frac{j}{m_1}(1-\frac{m_1}{m})^{1/2}]-(\log m)^{-2/3}$$

$$=c_1\log m-(\log m)^{-2/3}\geqslant c_0\log m\ .$$

where c_1, c_0 are positive constant. Hence

$$m\cdot\max_{m_0<j=m_1-(\log m)^2}P_{j,m}\leqslant m\cdot\int_{c_0\log m}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt$$

$$\sim\frac{m}{\sqrt{2\pi}}\frac{1}{c_0\log m}\exp\{-\frac{c_0(\log m)^2}{2}\}=\frac{m}{\sqrt{2\pi}}\frac{1}{c_0\log m}m^{-c_0^2(\log m)/2}\to 0,$$

which entails Lemma 2.

Lemma 3 Assume that $L(S_1,\cdots,S_n|c,\mu)$ be the likelihood ratio of $S_1,\cdots,S_n$ under $P_0^{(m)}$ relative to P_μ and $n=o(m^{1/2})$, $|\frac{c}{m}-\mu|=O(m^{-1/2})$. Then as $m\to\infty$,

$$L(S_1,\cdots,S_n|c,\mu)\to 1\quad \text{a.s.}\ P_\mu\ .$$

Proof $L(S_1,\cdots,S_n|c,\mu)$

$$=\frac{\varphi(S_1-\mu)\varphi(S_2-S_1-\mu)\cdots\varphi(S_n-S_{n-1}-\mu)\varphi(\frac{c-S_n-(m-n)\mu}{(m-n)^{1/2}})}{\varphi(S_1-\mu_1)\varphi(S_2-S_1-\mu)\cdots\varphi(S_n-S_{n-1}-\mu)\varphi(\frac{c-m\mu}{m^{1/2}})}$$

$$=(1-\frac{n}{m})^{-1/2}\exp\{-\frac{1}{2(1-n/m)}[\frac{n^2}{m}\frac{(S-n\mu)^2}{n^2}-2\frac{S_n-n\mu}{m^{1/2}}(\frac{c}{m^{1/2}}-\mu m^{1/2})$$

$$+n(\frac{c}{m}-\mu)^2]\}\to 0\quad \text{a.s.}\ P_\mu\ .$$

Lemma 4 As $m\to\infty$, for all (l,k) such that $l\geqslant m^{1/2}$, $n=k-l\geqslant m_0+m^{1/2}$, $\frac{n}{m}\to t'\in[t_0,t_1]$ and $x\leqslant(\log m)^{1/3}$, uniformly

$$(17)\qquad P_0^{(m)}\{S_j-S_i<b[(j-i)(1-\frac{j-i}{m})]^{1/2},\ \forall(i,j)\in J\,|S_k-S_l$$

$$=b[n(1-\frac{n}{m})]^{1/2}+x\}\sim P_{\mu_n}(\min_{j\geq 1}S_j>x)P_{\mu_n}(\min_{j\geq 1}S'_j+\min_{j\geq 0}S_j>x)$$

where $\mu_n=\frac{1}{2}\mu_1/[\frac{n}{m}(1-\frac{n}{m})]^{1/2}$, $\{S'_j,j\geqslant 1\}$ is an independent copy of $\{S_j,j\geqslant 1\}$, and

$$(18)\qquad J=J(l,k)=\{(i,j):0\leqslant i<j\leqslant m_1,\ j-i\geqslant m_0,\ j<k\ \text{or}\ j=k\ \text{and}\ i<1\}.$$

Proof. For $i,j\geqslant 1$, $n+i\leqslant m$, $n-j\geqslant 1$,

$$[n(1-\frac{n}{m})]^{1/2}+[(n-j+i)(1-\frac{n-j+i}{m})]^{1/2}>[(n-j)(1-\frac{n-j}{m})]^{1/2}$$
$$+[(n+i)(1-\frac{n+i}{m})]^{1/2}.$$

Hence the event on the left hand side of (17) equals

$$\{S_{k-j}-S_{l-i}<b[(n-j+i)(1-\frac{n-j+i}{m})]^{1/2},\ \forall (l-i,k-j)\in J\}$$
$$=\{S_k-S_{l-i}<b[(n+i)(1-\frac{n+i}{m})]^{1/2},\ \forall 1\leqslant i<(n-m_0)\wedge l\}$$
$$\cap\{S_{k-j}-S_{l-i}<b[(n-j+i)(1-\frac{n-j+i}{m})]^{1/2},\ \forall j-i<n-m_0, i\leqslant 0, j\geqslant 1\}$$

It follows Lemma 2 that

(19) $$P_0^{(m)}\{S_j-S_i<b[(j-i)(1-\frac{j-i}{m})]^{1/2},\ \forall (i,j)\in J\,|\,S_k-S_l=b[n(1-\frac{n}{m})]^{1/2}+x\}$$
$$=P_0^{(m)}\{S_k-S_{l-i}<b[(n+i)(1-\frac{n+i}{m})]^{1/2},\ 1\leqslant i<(\log m)^2;\ \text{and}$$
$$S_{k-j}-S_{l-i}<b[(n-j+i)(1-\frac{n-j+i}{m})]^{1/2},\ j-i<(\log m)^2,\ i\leqslant 0,$$
$$j\geqslant 1\,|\,S_k-S_l=b[n(1-\frac{n}{m})]^{1/2}+x\}+o(1)$$
$$=P_0^{(m)}\{S_{l-i}-S_l>x-\mu_1 i\frac{1-2n/m}{2[\frac{n}{m}(1-\frac{n}{m})]^{1/2}},\ 1\leqslant i<(\log m)^2;\ \text{and}$$
$$S_k-S_{k-j}+S_{l-i}-S_l>x+\mu_1(j-i)\frac{1-2n/m}{2[\frac{n}{m}(1-\frac{n}{m})]^{1/2}},\ 1\leqslant j-i<(\log m)^2,$$
$$i\leqslant 0, j\geqslant 1\,|\,S_k-S_l=b[n(1-\frac{n}{m})]^{1/2}+x\}+o(1)$$

As $m\to\infty$, it is easy via Lemma 3 to see that

$$\mathcal{L}\{S_{l-i}-S_l+\mu_1 i\frac{1-2n/m}{2[\frac{n}{m}(1-\frac{n}{m})]^{1/2}},1\leqslant i<(\log m)^2|S_m=0,$$
$$S_k-S_l=b[n(1-\frac{n}{m})]^{1/2}+x\}\Rightarrow\mathcal{L}\{S_i,\ i\geqslant 1\,|Y_1\sim N(\mu',1)\}$$
$$\mathcal{L}\{S_k-S_{k-j}-\mu_1 j\frac{1-2n/m}{2[\frac{n}{m}(1-\frac{n}{m})]^{1/2}},\ 1\leqslant j<(\log m)^2|S_m=0,$$
$$S_k-S_l=b[n(1-\frac{n}{m})]^{1/2}+x\}\Rightarrow\mathcal{L}\{S_j,\ j\geqslant 1\,|Y_1\sim N(\mu',1)\}$$
$$\mathcal{L}\{S_{l-i}S_l+\mu_1 i\frac{1-2n/m}{2[\frac{n}{m}(1-\frac{n}{m})]^{1/2}},\ -(\log m)^2<i\leqslant 0\,|S_m=0,$$

$$S_k - S_l = b[n(1-\frac{n}{m})]^{1/2} + x\} \Rightarrow \varphi\{S_i,\ i=0 | Y_1 \sim N(\mu', 1)\}$$

where $\mu' = \frac{1}{2}\mu_1/\sqrt{t'(1-t')}$, and asymptotically these three collections of random variables are stocastically independent. Hence the right hand side of (19) equals

$$P_\mu(\min_{j\geq 1} S_j > x) P_\mu(\min_{j\geq 1} S'_j + \min_{j\geq 0} S_j > x) + o(1)$$

$$\sim P_{\mu_n}(\min_{j\geq 1} S_j > x) P_{\mu_n}(\min_{j\geq 1} S'_j + \min_{j\geq 0} S_j > x).$$

The proof is completed.

Lemma 5 (Siegmund [3] Lemma 7)

Let $\{S'_n, n\geqslant 1\}$ be an independent copy of $\{S_n, n\geqslant 1\}$, $\mu>0$, then

$$\int_0^\infty e^{-2\mu x} P_\mu(\min_{j\geq 1} S_j > x) P_\mu(\min_{j\geq 1} S'_j + \min_{j\geq 0} S_j > x)\mathrm{d}x = 2\mu^3[\nu(2\mu)]^2$$

where $\nu(\cdot)$ is given in (8).

Proof of Theorem 1

$$(20)\qquad P_0^{(m)}(\tau \leqslant m_1) = \Big(\sum_{n=[m_0+\sqrt{m}]}^{m_1} \sum_{\substack{k-l=n \\ l\geq\sqrt{m},\ k\leq m_1}} + \sum_{n=[m_0+\sqrt{m}]}^{m_1} \sum_{\substack{k-l=n \\ l<\sqrt{m}}} + \sum_{n=m_0}^{[m_0+\sqrt{m}]-1} \sum_{k-l=n}\Big)$$

$$P_0^{(m)}\{S_k - S_l \geqslant b[n(1-\frac{n}{m})]^{1/2};\ S_j - S_i < b[(j-i)(1-\frac{j-i}{m})]^{1/2},$$

$$\forall (i,j)\in J(l,k)\} \triangleq P_1 + P_2 + P_3.$$

From Lemma 1,4,5

$$P_1 = \sum_{n=[m_0+m^{1/2}]}^{m_1} \sum_{\substack{k-l=n \\ l\geq m^{1/2},\ k\leq m_1}} \int_0^\infty P_0^{(m)}\{S_k - S_l \in b[n(1-\frac{n}{m})]^{1/2} + \mathrm{d}x\}$$

$$\times P_0^{(m)}\{S_j - S_i < b[(j-i)(1-\frac{j-i}{m})]^{1/2},\ \forall (i,j)\in J(l,k)$$

$$| S_k - S_l = b[n(1-\frac{n}{m})]^{1/2} + x\}$$

$$\sim \sum_{n=[m_0+m^{1/2}]}^{m_1} (m_1 - n)[2\pi n(1-\frac{n}{m})]^{-1/2} e^{-\frac{1}{2}b^2} \int_0^\infty e^{-2\mu_n x}$$

$$\times P_{\mu_n}(\min_{j\geq 1} S_j > x) P_{\mu_n}(\min_{j\geq 1} S'_j + \min_{j\geq 0} S_j > x)\mathrm{d}x$$

$$= \frac{m}{4} b\varphi(b) \sum_{n=[m_0+\sqrt{m}]}^{m_1} \mu_1^2 (t_1 - \frac{n}{m})[\frac{n}{m}(1-\frac{n}{m})]^{-2} \nu^2(2\mu_n)\frac{1}{m}$$

$$\sim \frac{m}{4} b\varphi(b) \int_{t_0}^{t_1} \frac{t_1 - t}{[t(1-t)]^2} \mu_1^2 \nu^2 \Big(\frac{\mu_1^2}{[t(1-t)]^{1/2}}\Big)\mathrm{d}t$$

Let $x = \sqrt{\frac{1-t}{t}}$ in the right hand side of above expression, it becomes

$$\frac{m}{2} b\varphi(b) \int_{\mu_1(t_1^{-1}-1)^{1/2}}^{\mu_1(t_0^{-1}-1)^{1/2}} \frac{1}{x}(t_1 - \mu_1^2 \frac{1-t_1}{x^2})(x^2 + \mu_1^2)\nu^2(x + \frac{\mu_1^2}{x})\mathrm{d}x$$

To complete the proof of the theorem we need only to verify that

(21) $$P_2 = o(m^{3/2} e^{-\frac{1}{2}b^2});$$

(22) $$P_3 = o(m^{3/2} e^{-\frac{1}{2}b^2}).$$

From Lemma 1

$$P_2 \leqslant \sqrt{m} \sum_{n=m_0}^{m_1} P_0^{(m)}\{S_n \geqslant b[n(1-\frac{n}{m})]^{1/2}\}$$

$$\leqslant m^{1/2} \sum_{n=m_0}^{m_1} e^{-\frac{1}{2}b^2}[2\pi n(1-\frac{n}{m})]^{-1/2} \int_0^\infty \exp\{-[\frac{n}{m}(1-\frac{n}{m})]^{-1/2}\mu_1 x\} dx$$

$$+ m^{1/2}(m_1 - m_0)\cdot o({}^{-1/2} e^{-\frac{1}{2}b^2}) = O(me^{-\frac{1}{2}b^2}).$$

which proves (21). Similaly

$$P_3 \leqslant m \sum_{n=m_0}^{[m_0+\sqrt{m}]} P_0^{(m)}\{S_n \geqslant b[n(1-\frac{n}{m})]^{1/2}\} = O(me^{-\frac{1}{2}b^2}).$$

Hence (22) is also valid.

Proof of Corollary 1 To expess $P_0^{(m)}(T \leqslant m_1)$ in the form of (20). From (21), (2), and $P_0^{(m)}\{|S_n| \geqslant b[n(1-\frac{n}{m})]^{1/2}\} = 2P_0^{(m)}\{S_n \geqslant b[n(1-\frac{n}{m})]^{1/2}\}$, it is easy to see that if

(23) $$P_0^{(m)}\{|S_k - S_l| \geqslant b[n(1-\frac{n}{m})]^{1/2};\ |S_j - S_i| \quad b[(j-i)(1-\frac{j-i}{m})]^{1/2}, \text{for all } (i,j)\in J(l,k)\}$$

$$\sim 2P_0^{(m)}\{S_k - S_l \geqslant b[n(1-\frac{n}{m})]^{1/2}; S_j - S_i < b[(j-i)(1-\frac{j-i}{m})]^{1/2}, \text{for all } (i,j)\in J(l,k)\}$$

then

$$P_0^{(m)}(T \leqslant m_1) \sim 2P_0^{(m)}\ (\tau \leqslant m_1)$$

which entails Corollary 1. Thus we only need to prove (23).

(24) RHS (23)—LHS (23)

$$= 2P_0^{(m)}\{S_k - S_l \geqslant b[n(1-\frac{n}{m})]^{1/2};\ S_j - S_i < b[(j-i)(1-\frac{j-i}{m})]^{1/2},$$

$$\text{for all } (i,j)\in J(l,k);\ S_{j_0} - S_{i_0} \leqslant -b[(j_0-i_0)(1-\frac{j_0-i_0}{m})]^{1/2},$$

$$\text{for some } (i_0, j_0)\in J(l,k)\} \leqslant 2P_0^{(m)}\{S_k - S_l \geqslant b[n(1-\frac{n}{m})]^{1/2};$$

$$S_j - S_i \leqslant -b[(j-i)(1-\frac{j-i}{m})]^{1/2} \text{ for some } (i,j)\in J(l,k)\}$$

$$= 2\int_0^\infty P_0^{(m)}\{S_j - S_i \leqslant -b[(j-i)(1-\frac{j-i}{m})]^{1/2} \text{ for some}$$

$$(i,j)\in J(l,k)\ |S_k - S_l = b[n(1-\frac{n}{m})]^{1/2} + x\}$$

$$\times P_0^{(m)}\{S_n \in b[n(1-\frac{n}{m})]^{1/2} + dx\}$$

Use the method to prove Lemma 2, one can show

(25) $P_0^{(m)}\{S_j - S_i > -b[(j-i)(1-\frac{j-i}{m})]^{1/2}$ for all $(i,j)\epsilon J(l,k)$

$$|S_k - S_l = b[n(1-\frac{n}{m})]^{1/2} + x\}$$

$$= P_0\{S_i < b[n(1-\frac{n}{m})]^{1/2} + x + b[(n-i)(1-\frac{n-i}{m})]^{1/2} \text{ for all}$$

$$1 \leqslant i < (n-m_0) \wedge 1\, |S_{m-n} = b[n(1-\frac{n}{m})]^{1/2} + x\}$$

$$\times P_0\{S_j < b[n(1-\frac{n}{m})]^{1/2} + x + b[(n-j)(1-\frac{n-j}{m})]^{1/2} \text{ for all}$$

$$1 \leqslant j < n - m_0 |S_n = b[n(1-\frac{n}{m})]^{1/2} + x\} \to 1$$

uniformly in all $x \leqslant (\log m)^{1/3}$ and (l,k) such that $l \geqslant m^{1/2}$, $k \leqslant m_1$, $n = k-1 \geqslant m_0 + m^{1/2}$. It follows (25) and Lemma 1 that

$$\text{RHS } (24) = o(m^{-\frac{1}{2}} e^{-\frac{1}{2}b^2})$$

From Lemma 2,5, it is easy to see that

$$\text{RHS } (23) = O(m^{-\frac{1}{2}} e^{-\frac{1}{2}b^2})$$

Hence (24) entails (23) valid.

5. Proof of Theorem 2

In this Section, we always assume that $b = \mu_1 m^{1/2}$, $m_0 = t_0 m$, $m_1 = t_1 m$, $\xi = \xi_0 m$, $\mu_1 > 0, 0 < t_0 < t_1 < 1$, and

(26) $$t^* = \frac{\xi_0^2}{\xi_0^2 + (1-t_1)\mu_1^2}, \qquad \mu^* = \frac{1}{2}[\frac{1-t}{\xi_0}\mu_1^2 + \frac{\xi_0}{1-t_1}]$$

(27) $$D = D(\xi_0) = \{(l,k): 1 \geqslant m^{1/2},\ k < m_1,\ |k - l - mt^*| \leqslant m^{1/12}\}$$

We also assume that

$$\xi_0 \epsilon (\mu_1 (1-t_1)\sqrt{\frac{t_0}{1-t_0}}, \ \mu_1 \sqrt{t_1(1-t_1)}).$$

except in the proof of Corollary 2, where $|\xi_0|$ will substitute ξ_0 in above expression.

Lemma 6 As $m \to \infty$,

(i) for $|n - mt^*| \leqslant m^{7/12}$, $x \leqslant (\log m)^{1/3}$ uniformly

(28) $$P_\xi^{(m)}\{S_n \epsilon b[n(1-\frac{n}{m})]^{1/2} + \mathrm{d}x\} \sim m^{-\frac{1}{2}} \exp\{-\frac{m}{2}(\mu_1^2 - \frac{\xi_0^2}{t_1(1-t_1)})\} \sqrt{\frac{t_1}{t^*(t_1 - t^*)}}$$

$$\times \varphi[\sqrt{\frac{m}{1-n/m_1}}(\sqrt{\frac{n}{m_1}}(1-t_1)\mu_1 - \sqrt{\frac{1-n/m}{t_1(1-t_1)}}\xi_0)] e^{-2\mu^* x}$$

where $\varphi(x) = (2\pi)^{-1/2} \exp(-\frac{1}{2}x^2)$;

(ii) for $|n - mt^*| \leqslant m^{7/12}$, $x > m^{1/3}$ uniformly

(29) $$P_\xi^{(m_1)}\{S_n \geqslant b[n(1-\frac{n}{m})]^{1/2} + x\} = o(m^{-2} \exp\{-\frac{m}{2}(\mu_1^2 - \frac{\xi_0^2}{t_1(1-t_1)})\});$$

(iii) for $m_0 \leqslant n \leqslant m_1$, $|n - t^* m| > m^{7/12}$ uniformly

$$(30)\quad P_\xi^{(m_1)}\{S_n \geqslant b[n(1-\frac{n}{m})]^{1/2}\} = o\{m^{-2}\exp\{-\frac{m}{2}(\mu_1^2 - \frac{\xi_0^2}{t_1(1-t_1)})\}\}.$$

Lemma 6 follows $\varphi(S_n|S_{m_1}=\xi) = N(n\xi_0/t_1,\ n(1-\frac{n}{m_1}))$, some standard estimates, and

$$\frac{1}{n(1-n/m_1)}[b\sqrt{n(1-\frac{n}{m})} - \frac{n\xi_0}{t_1}] \to 2\mu^*$$

for $n \sim mt^*$.

Lemma 7 As $m \to \infty$, uniformly for $x \leqslant (\log m)^{1/3}$ and $|n - mt^*| \leqslant m^{7/12}$

$$P_0\{S_{m_1-j} < b[j(1-\frac{j}{m})]^{1/2} - \xi,\ \forall n + (\log m)^2 \leqslant j < n + (n-m_0)\wedge l$$

$$|S_{m_1-n} = b[n(1-\frac{n}{m})]^{1/2} + x - \xi\} \to 1.$$

Proof Similarly to proof of Lemma 2, it only needs to show

$$(31)\quad \min_{n+(\log m)^2 \leq j < n+(n-m_0)\wedge l} c'_{j,m} = o(\log m)$$

where

$$c'_{j,m} = [(m_1 - j)(1 - \frac{m_1 - j}{m_1 - n})]^{-\frac{1}{2}}\{b[j(1-\frac{j}{m})]^{\frac{1}{2}} - \xi - \frac{m_1 - j}{m_1 - n}[b[n(1-\frac{n}{m})]^{\frac{1}{2}} + x - \xi]\}.$$

Let $t_j = \frac{j}{m}$, $t_n = \frac{n}{m}$. Then $|t_n - t^*| \leqslant m^{-5/12}$. From (26), $\xi_0 = \mu_1 t^*(1-t_1)/\sqrt{t^*(1-t^*)}$. Hence,

$$(32)\quad c'_{j,m} \sim \mu_1 m^{\frac{1}{2}}\sqrt{t_j - t_n}\,\frac{t_1 t_j t^* + d_{j,m}}{\sqrt{(t_1 - t_j)(t_1 - t^*)t^*(1-t^*)}\,[(t_1 - t^*)\sqrt{t_j(1-t_j)} + (t_1 - t_j)\sqrt{t^*(1-t^*)}]}$$

where

$$d_{j,m} = (t_1 - t^*)[t_1(1-t_j)\sqrt{t^*(1-t^*)} - t^*(1-t_1)\sqrt{t_j(1-t_j)}\}.$$

Since $t_1 \geqslant t_j$, $t_1 \geqslant t^*$, and consequently $t_1\sqrt{t^*} \geqslant t^*\sqrt{t_j}$ for $n + (\log m)^2) \leqslant j < n + (n - m_0)$ $\wedge l$,

$$d_{j,m} \geqslant (t_1 - t^*)\sqrt{1-t^*}\,[t_1\sqrt{t^*}(1-t_j) - t^*\sqrt{t_j}(1-t_1)\sqrt{\frac{1-t_n-\frac{1}{m}(\log m)^2}{1-t^*}}]$$

$$\sim (t_1 - t^*)\sqrt{1-t^*}\,[t_1\sqrt{t^*}(1-t_j) - t^*\sqrt{t_j}(1-t_1)] \geqslant 0$$

On the other hand

$$\min_{n+(\log m)^2 \leq j < n+(n-m_0)\wedge l} m^{1/2}\sqrt{t_j - t_n} = \log m$$

Hence (31) follows (32).

Lemma 8 As $m \to \infty$, for all $(l,k) \in D$ and $x \leqslant m^{1/3}$ uoiformly

$$P_\xi^{(m_1)}\{S_j - S_i < b[(j-i)(1-\frac{j-i}{m})]^{\frac{1}{2}},\ \forall (i,j) \in J | S_k - S_l = b[(k-l)(1-\frac{k-l}{m})]^{\frac{1}{2}} + x\}$$

$$\to P_{\mu^*}(\min_{j\geq 1} S_j > x) P_{\mu^*}(\min_{j\geq 1} S'_j + \min_{j\geq 0} S_j > x)$$

where $\{S'_j,\ j \geqslant 1\}$ is an independent copy of $\{S_j,\ j \geqslant 1\}$.

Lemma 8 can be shown in terms of Lemma 7, 2, 3. The detailed proof is similar to the proof of Lemma 4.

Proof of Theorem 2

$$P_\xi^{(m_1)}(\tau<m_1)=\Big(\sum_{(l,k)\in D}+\sum_{(l,k)\in D}\Big)P_\xi^{(m_1)}\{S_k-S_l\geqslant b[n(1-\frac{n}{m})]^{1/2},$$

$$S_j-S_i<b[(j-i)(1-\frac{j-i}{m})]^{1/2},\ \forall(i,j)\in J(l,k)\}\widehat{=}P_1+P_2$$

From Lemma 8, Lemma 5 and (28), (29)

$$p_1=\sum_{(l,k)\in D}\int_0^\infty P_\xi^{(m_1)}\{S_k-S_l\in b[n(1-\frac{n}{m})]^{1/2}+\mathrm{d}x\}P_\xi^{(m_1)}\{S_j-S_i$$

$$<b[(j-i)(1-\frac{j-i}{m})]^{1/2},\ \forall(i,j)\in J(l,k)\,|S_k-S_l=b[n(1-\frac{n}{m})]^{1/2}+x\}$$

$$\sim m(t_1-t^*)\exp\{-\frac{m}{2}(\mu_1^2-\frac{\xi_0^2}{t_1(1-t_1)})\}2\mu^{*3}v^2(2\mu^*)$$

$$\times\sum_{n=[mt^*-m^{7/12}]}^{[mt^*+m^{7/12}]}[\frac{mt_1}{(t-t^*)t^*}]^{1/2}\varphi[\sqrt{\frac{m}{1-n/m_1}}(\sqrt{\frac{n}{m_1}}(1-t_1)\mu_1-\sqrt{\frac{1-n/m}{t_1(1-t_1)}}\xi_0)]\frac{1}{m}$$

The sum in the right hand side of above expression converges to

$$2[\sqrt{\frac{1-t_1}{t_1}}\mu_1+\frac{\xi_0}{t_1(1-t_1)}\sqrt{\frac{t^*}{1-t^*}}]^{-1}$$

By (26),

$$P_1\quad\frac{m}{2}\exp\{-\frac{m}{2}(\mu_1^2-\frac{\xi_0^2}{t_1(1-t_1)})\}\sqrt{t_1(1-t_1)}\ \frac{\mu_1}{\xi_0}[\frac{t_1(1-t_1)}{\xi_0}\mu_1^2-\xi_0]$$

$$\times(\frac{1-t_1}{\xi_0}\mu_1^2+\frac{\xi_0}{1-t_1})v^2(\frac{1-t_1}{\xi_0}\mu_1^2+\frac{\xi_0}{1-t_1})$$

On the other hand, it follows (30) that

$$P_2=o(m\exp\{-\frac{m}{2}(\mu_1^2-\frac{\xi_0^2}{t_1(1-t_1)})\})$$

which completes the proof of Theorem 2.

Proof of Corollary 2 Now $|\xi_0|\in(\mu_1(1-t_1)\sqrt{\frac{t_0}{1-t_0}},\ \mu_1\sqrt{t_1(1-t_1)})$. There is no loss of generality to assume $\xi_0>0$, Then

$$P_\xi^{(m_1)}(T<m_1)-P_\xi^{(m_1)}(\tau<m_1)$$

$$=P_\xi^{(m_1)}\{\min_{1\leq l<k\leq m_1,\ k-l\geq m_0}\frac{S_k-S_l}{[(k-l)(1-\frac{k-l}{m})]^{1/2}}\leqslant-b,\ \tau\geqslant m_1\}$$

$$\leqslant P_\xi^{(m_1)}\{\min_{1\leq l<k\leq m_1,\ k-l\geq m_0}(S_k-S_l)/\sqrt{(k-l)(1-\frac{k-l}{m})}\leqslant-b\}=P_{-\xi}^{(m_1)}(\tau<m_1)$$

$$\leqslant m\sum_{n=m_0}^{m_1-1}P_{-\xi}^{(m_1)}(S_n\geqslant b[n(1-\frac{n}{m})]^{1/2})\leqslant c\cdot\exp\{-\frac{m}{2}(\mu_1^2-\frac{\xi_0^2}{t_1(1-t_1)})\}$$

$$\times m\sum_{n=m_0}^{m_1-1}\varphi[\sqrt{\frac{m}{1-n/m_1}}(\sqrt{\frac{n}{m_1}}(1-t_1)\mu_1+\sqrt{\frac{1-n/m}{t_1(1-t_1)}}\xi_0)]$$

$$=o(m\exp\{-\frac{m}{2}(\mu_1^2-\frac{\xi_0^2}{t_1(1-t_1)})\}).$$

Hence $P_{\xi}^{(m_1)}(T<m_1) \sim P_{\xi}^{(m_1)}(\tau<m_1)$, which entails Corollary 2.

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