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Approximation of Meromorphic Function by Rational Interpolating Functions in the Complex Plane*

Gu Xiaoying

(China University of Geosiences, Beijing)

Abstract In this paper for a Jordan domaon D with $1+\varepsilon$ $(\varepsilon>0)$ smoothness the degree of approximation in $L^p(\partial D)$, $0<p<+\infty$, of rational interpolating functions at the Fejer points of meromorphic functions with finite poles in D is obtained.

I . Introduction

Let D be a Jordan domain with its boundary Γ and $z=\Psi(w)$ is the function conformaly mapping $|w|>1$ onto the complement of $\overline{D}$ with the conditions $\Psi(\infty)=\infty$, $\Psi'(\infty)>0$. it is well known that function $z=\Psi(w)$ can be extended into the boundary $|w|=1$ and realizes the one to one continuous map between the $|w|=1$ and Γ. The points:

$$z_{nk}=\Psi(\exp(\frac{2ki}{n+1})), \qquad (k=0,1,\cdots,n) \tag{1.1}$$

are called the Fejer interpolation points .

In 1988 Shen and Zhong obtained the following theorem concering the approximation by interpolating polynomials at Fejer points (see [1]):

Theorem A Let D be a Jordan domain with its boundary Γ and has $1+\varepsilon$ smoothness, $\varepsilon>0$, it means that Γ has a parametric equation $z=z(t)$, $z'(t)\in\mathrm{Lip}\,\varepsilon$. Denote by $A(\overline{D})$ the class of functions, analytic in D and continuous on $\overline{D}$. Then for each p, $0<p<+\infty$ and each natural number n.

$$\{\int_\Gamma|f(z)-L_n(f,z)|^p|dz|\}^{1/p}<C\omega(f,\frac{1}{n}) \tag{1.2}$$

where C is a constant, $L_n(f,z)$ is the Lagrange interpolating polynomials of degree at most n of $f(z)\in A(\overline{D})$ at the Fejer points $\{z_{nk}\}$, $0<k<n$ and $\omega(f,\delta)$ is the modulus of continuity of $f(z)$ on $\overline{D}$:

$$\omega(f,\delta)=\sup_{\substack{|h|<\delta \\ z\in\overline{D},\ z+h\in\overline{D}}}|f(z+h)-f(z)| \tag{1.3}$$

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Here and after we denote by C constant, no matter how large it is.

From the procedure of proof we can see that if we use the equivalent property between the modulus of continuity of $g(w)\in A(|w|\leqslant 1)$ on $|w|\leqslant 1$ and $|w|=1$ and the property of mapping function

$$0<C_1\leqslant|\varphi'(w)|\leqslant C_2\ ,\quad |w|\leqslant 1 \tag{1.4}$$

where $z=\varphi(w)$)is the function conformally maping $|w|<1$ onto D, then the modulus of continuity defined by (1.4) is equivalent to the modulus of continuity on the Γ:

$$\omega_1(f,\delta)=\sup_{\substack{|h|\leqslant\delta\\ z\in\Gamma,\ z+h\in\Gamma}}|f(z+h)-f(z)| \tag{1.5}$$

(see [2]). Hence we can substitute the right-hand side of (1.2) by $C\omega_1(f,\frac{1}{n})$.

It is natural to generalize Theorem A to the meromorphic functions in the domain D.

We denote by $M_v(\overline{D})$ the class of functions analytic in D except for v poles $\{a_i\}$, $1\leqslant i\leqslant v$ and continuous on $\overline{D}$ except for these points.

Let $\{\beta_j\}$, $1\leqslant j\leqslant m$ be the different points in $\{a_i\}$, $1\leqslant i\leqslant v$ and the number of appearence of β_j in $\{a_i\}$, $1\leqslant i\leqslant v$, is denoted by p_j. It is obvious that

$$\sum_{k=1}^{m}p_k=v. \tag{1.6}$$

Let

$$Q_1(z)=(z-\beta_1),\cdots,Q_{p_1}(z)=(z-\beta_1)^{p_1}$$

$$Q_{p_1+\cdots+p_i+j}(z)=\left(\prod_{k=1}^{i}(z-\beta_k)^{p_k}\right)(z-\beta_{i+1})^j,\ (1\leqslant i\leqslant n-1,0\leqslant j\leqslant p_{i+1}). \tag{1.7}$$

Hence from (1.6) and (1.7) we have

$$Q_v(z)=\prod_{k=1}^{m}(z-\beta_k)^{p_k}. \tag{1.8}$$

We denote by $r_{n,v}(z)=\dfrac{p_n(z)}{q_v(z)}$ the rational function, the numerator $p_n(z)$ of which is a polynomial of degree at most n, and the denominator $q_v(z)$ is a polynomial of degree at most v.

In this paper we will prove the following theorem.

Theorem . Let the boundary Γ of simply connected domain D is belonging to class $C^{1+\varepsilon}$, $\varepsilon>0$, $f(z)\in A(\overline{D})$. Then

1°. For sufficient large n there exists a rational function $r_{n,v}(z)=\dfrac{p_n(z)}{q_v(z)}$, which interpolates $f(z)$ at $(n+v+1)$ Fejer points $z_{n+v,k}$, $0\leqslant k\leqslant n+v$ (see (1.1)), i.e.

$$r_{n,v}(z_{n+v,k})=f(z_{n+v,k}),\ 0\leqslant k\leqslant n+v\ ; \tag{1.9}$$

2°. For any compact in the complex plane

$$\lim_{n\to+\infty} q_\nu(z)=Q_\nu(z) \tag{1.10}$$

is valid uniformly;

For sufficient large n, $r_{n,\nu}(z)$ has ν poles $a_k^{(n)}$, $1<k<\nu$ such that

$$\lim_{n\to+\infty} a_k^{(n)}=a_k,\ 1<k<\nu; \tag{1.11}$$

3°. For any p, $0<p<+\infty$ and natural number n we have

$$\{\int_\Gamma |f(z)-r_{n,\nu}(z)|^p|dz|\}^{1/p}< C\omega(f,\frac{1}{n}); \tag{1.12}$$

4°.

$$\lim_{n\to+\infty} r_{n,\nu}(z)=f(z) \tag{1.13}$$

is valid uniformly inside D except for $\{\beta_j\}$, $1<j<m$.

Remark If D is the unite circle $|z|<1$; Then 1°. for $p=2$ (1.13) was obtained in [4]; 2°. for any p, $0<p<+\infty$

$$\lim_{n\to+\infty}\int_{|z|=1}|f(z)-r_{n,\nu}(z)|^p|dz|=0.$$

was obtained in [5]. It is obvious that all these cases are our special cases.

2. Preliminary lemmas

Lemma 1 The modulus of coninuity of functions $f(z)$ and $Q_i(z)f(z)$, $1<i<\nu$, defined by (1.7) are equivalent.

In fact it is easy to get Lemma 1 only if we use

$$0<C_3<|Q_i(z)|<C_4,\ z\in\Gamma,\ 1<i<\nu$$

and the definition of modulus of continuity (1.5).

We denote by $E_p(D)$ the class of functions $f(z)$ analytic in D and satisfying

$$\lim_{n\to+\infty}|\int_{\Gamma_n}|f(z)|^p|dz|)^{1/p}<+\infty,$$

(see [3]), where Γ_n is the equipotational line $\Psi(|w|=1-\frac{1}{n})$. It is known that if $f(z)\in E_p(D)$ and the boundary of D is a closed Jordan rectifiable curve, then $f(z)$ has angular boundary values on Γ almost everywhere and

$$(\int_\Gamma|f(\zeta)|^p|d\zeta|)^{1/p}=\lim_{n\to+\infty}(\int_{\Gamma_n}|f(z)|^p|dz|)^{1/p}<+\infty.$$

Lemma 2 Let the boundary Γ of D is belonging $C^{1+\varepsilon}$, $\varepsilon>0$, $f(z)\in E_p(D)$, $0<p<+\infty$. Then for any compact $F\subset D$ we have

$$|f(z)|<C(F)\{\int_\Gamma|f(\zeta)|^p|d\zeta|\}^{1/p},\ z\in F\subset D, \tag{2.1}$$

where $C(F)$ is a constant, only depending on F.

Proof For $p>1$ using the Cauchy formula (see [3])

$$f(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta,\quad z\in F\subset D,$$

inequality

$$\operatorname*{dist}_{\zeta\in\Gamma,\, z\in F}(\zeta, z)>0$$

and the Hölder inequality we can get (2.1) immediately.

For $0<p<1$ and $F\subset D$ there exists an equipotational line such that F is contained inside $\Gamma_{1-\frac{\tau}{m}}$. Thus we have

$$f(z)=\frac{1}{2\pi i}\int_{\Gamma_{1-\frac{1}{m}}}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta,\quad z\in F\subset\Gamma_{1-\frac{1}{m}}. \tag{2.2}$$

Consequently by using (see [3])

$$0<C_3<\left|\frac{\Psi(w)-\Psi(\tau)}{w-\tau}\right|<C_4,\quad |w|<1,\quad |\tau|<1,$$

we know

$$\operatorname*{dist}_{\zeta\in\Gamma_{1-\frac{1}{m}},\, z\in\Gamma_{1-\frac{\tau}{m}}}(\zeta, z)\geqslant C>0.$$

Hence by applying (1.4), and (2.2) we get

$$|f(z)|\leqslant C(F)\int_{1-\frac{1}{m}}|f(\zeta)||\mathrm{d}\zeta|\leqslant C(F)\int_{|\tau|=1-\frac{1}{m}}|f[\Psi(\tau)]||\mathrm{d}\tau|,\quad z=\Psi(\omega).$$

By using inequality between different norms (see formula (3) in [6], § 3) from (2.3) it follows

$$|f(z)|\leqslant C(F)m^{\frac{1}{p}-1}\{\int_{|\tau|=1-\frac{1}{2m}}|f[\Psi(\tau)]|^p|\mathrm{d}\tau|\}^{\frac{1}{p}}$$
$$\leqslant C(F)\{\int_{|\tau|=1}|f[\Psi(\tau)]|^p|\mathrm{d}\tau|\}^{\frac{1}{p}}\leqslant C(F)\{\int_{\Gamma}|f(\zeta)|^p|\mathrm{d}\zeta|\}^{\frac{1}{p}},\quad z\in F,$$

where the last inequality is obtained in virtue of (1.4).

This completes the proof of Lemma 2.

Lemma 3 For sufficient large n there exists a polynomial of degree ν:

$$q_\nu(z)=Q_\nu(z)+\sum_{k=1}^{\nu}a_k^{(n)}Q_{k-1}(z),\ Q_0(z)=1, \tag{2.4}$$

such that the Lagrange interpolating polynomial $L_{n+\nu}(q_\nu Q_\nu f;z)$ of degree at most $n+\nu$ of $q_\nu(z)Q_\nu(z)f(z)$ at $n+\nu+1$ Fejer points (1.1) can be divided by $Q_\nu(z)$ and

$$\lim_{n\to+\infty}q_\nu(z)=Q_\nu(z) \tag{2.5}$$

is valid uniformly on any compact in the complex plane.

Proof We have

$$L_{n+\nu}(q_\nu Q_\nu f;z)=L_{n+\nu}(Q_\nu^2 f;z)+\sum_{k=1}^{\nu}a_k^{(\nu)}L_{n+\nu}(Q_{k-1}Q_k f;z). \tag{2.6}$$

Hence if we choose $a_k^{(n)}$ such that

$$L_{n+\nu}^{(j)}(q_\nu Q_\nu f;\beta_k)=0,\quad j=0,1,\cdots,p_k-1, \tag{2.7}$$

so $L_{n+\nu}(q_\nu Q_\nu f; z)$ can be divided by $Q_\nu(z)$. Thus from (2.6) and (2.7) we get

$$\sum_{k=1}^{\nu} a_k^{(n)} L_{n+\nu}(Q_{k-1}Q_\nu f; \beta_1) = -L_{n+\nu}(Q_\nu^2 f; \beta_1)$$
$$\vdots$$
$$\sum_{k=1}^{\nu} a_k^{(n)} L_{n+\nu}^{(p_1-1)}(Q_{k-1}Q_\nu f; \beta_1) = -L_{n+\nu}^{(p_1-1)}(Q_\nu^2 f; \beta_1),$$
$$\sum_{k=1}^{\nu} a_k^{(n)} L_{n+\nu}(Q_{k-1}Q_\nu f; \beta_2) = -L_{n+\nu}(Q_\nu^2 f; \beta_2);$$
$$\vdots$$
$$\sum_{k=1}^{\nu} a_k^{(n)} L_{n+\nu}^{(p_2-1)}(Q_{k-1}Q_\nu f; \beta_2) = -L_{n+\nu}^{(p_2-1)}(Q_\nu^2 f; \beta_2),$$
$$\sum_{k=1}^{\nu} a_k^{(n)} L_{n+\nu}(Q_{k-1}Q_\nu f; \beta_m) = -L_{n+\nu}(Q_\nu^2 f; \beta_m),$$
$$\vdots$$
$$\sum_{k=1}^{\nu} a_k^{(n)} L_{n+\nu}^{(p_m-1)}(Q_{k-1}Q_\nu f; \beta_m) = -L_{n+\nu}^{(p_m-1)}(Q_\nu^2 f; \beta_m).$$

From Theorem A, Lemmas 1 and 2 it follows that the coefficients of linear equation (2.8) and the right-hand side have the limits.

$$\lim_{n\to+\infty} -L_{n+\nu}^{(s)}(Q_\nu^2 f; \beta_j) = -[(Q_\nu^2 f)^{(s)}(\beta_j)] = 0 \qquad (2.9)$$
$$(j=1,2,\cdots,m;\ s=0,1,\cdots,p_j-1)$$

and

$$\lim_{n\to+\infty} L_{n+\nu}^{(s)}(Q_{k-1}Q_\nu f; \beta_j) = (Q_{k-1}Q_\nu f)^{(s)}(\beta_j). \qquad (2.4)$$
$$(k=1,2,\cdots,m;\ j=1,2,\cdots,m,\ s=0,1,\cdots,p_j-1)$$

It is obvious that for $k-1 < \sum_{i=1}^{j-1} p_i, 0 < s < p_j - 1$ and $\sum_{i=1}^{j-1} p_i < k-1 < \sum_{i=1}^{j} p_i$, $s > k-1-\sum_{i=1}^{j-1} p_i$, $(Q_{k-1}Q_\nu f)^{(s)}(\beta_j) \neq 0$; and for $\sum_{i=1}^{j-1} p_i < k-1 < \sum_{i=1}^{j} p_i$, $s < k-1-\sum_{i=1}^{j-1} p_i$ and $k-1 > \sum_{i=1}^{j} p_i$, $(Q_{k-1}Q_\nu f)^{(s)}(\beta_j) = 0$.

It follows that when n theds to $+\infty$, the determinant of coefficients of linear equation (2.8) tends to

$$\prod_{j=1}^{m} \prod_{k=\sum_{i=1}^{j-1} p_i+1}^{\sum_{i=1}^{j} p_i} (Q_{k-1}Q_\nu f)^{(s-1-\sum_{i=1}^{j-1} p_i)}(\beta_j) \neq 0. \qquad (2.1)$$

Thus according to the Cramer criteria we know that for sufficient large n the equation (2.8) has a solution $a_k^{(n)}$, $1 < k < \nu$ and from (2.9) and (2.11) it follows

$$\lim_{n\to+\infty} a_k^{(n)} = 0,\ 1 < k < \nu.$$

Hence from (2.4) we know

$$\lim_{n\to+\infty} q_\nu(z) = Q_\nu(z)$$

is valid uniformly on any compact in the complex plane.

Lemma 3 is proved completely.

3. Proof of the main theorem

At first from Lemma 3 we know that rational function

$$r_{n,v}(z)=\frac{L_{n+v}(q_vQ_v f; z)}{q_vQ_v}$$

has its numerator as a polynomial

$$P_n(z)=\frac{L_{n+v}(q_vQ_v f; z)}{Q_v(z)}$$

of degree at most n and from (2.5) its denominator $q_v(z)$ is a polynomial of degree at most v and doesn't have any zeros on Γ. Consequently, $r_{n,v}(z)$ is the rational interlolating function at $n+v+1$ Fejer points (1.1). This completes the proof of 1° in the Theorem.

From Lemma 3 we can get (1.10) in 2° of the Theorem.

Besides, from Theorem A we have

$$\{\int_\Gamma |L_{n+v}(Q_{k-1}Q_v f; z)-Q_{k-1}Q_v f||dz|\}^{1/p}\begin{cases}\leqslant C\omega_1(f;\frac{1}{n+v})\\ \leqslant C\omega_1(f,\frac{1}{n})\end{cases}\tag{3.1}$$

where $1\leqslant k\leqslant v+1$.

Thus for $p\geqslant 1$ using Minkowski equality, and for $0<p<1$ using the following equality:

$$(|a|+|b|)^p\leqslant |a|^p+|b|^p$$

and applying (3.1) we get

$$\begin{aligned}&\{\int_\Gamma |L_{n+v}(q_vQ_v f; z)-q_vQ_v f|^p|dz|\}^{1/p}\\ &\leqslant C\{\int_\Gamma |L_{n+v}(Q_v^2 f; z)-Q_v^2 f|^p|dz|\}^{1/p}\\ &\quad +C\sum_{k=1}^{v}|a_k^{(v)}|\{\int_\Gamma |L_{n+v}(Q_{k-1}Q_v f; z)-Q_{k-1}Q_v f|^p|dz|\}^{1/p}\\ &\leqslant C\omega(f,\frac{1}{n}).\end{aligned}\tag{3.2}$$

Hence by using (2.5) and (3.2) we have

$$\{\int_\Gamma\left|\frac{L_{n+v}(q_vQ_v f; z)}{q_vQ_v}-f\right|^p|dz|\}^{1/p}\leqslant C\omega(f,\frac{1}{n}).\tag{3.3}$$

This proved the 3° in the Theorem.

From (3.2) and (2.5) we can easily get

$$\begin{aligned}&\{\int_\Gamma |L_{n+v}(q_vQ_v f; z)-Q_v^2 f|^p|dz|\}^{1/p}\\ &\leqslant C\{\int_\Gamma |L_{n+v}(q_vQ_v f; z)-q_vQ_v f|^p|dz|\}^{1/p}\end{aligned}$$

$$+C\{\int_\Gamma |q_\nu - Q_\nu|^p |Q_\nu f|^p |dz|\}^{1/p} \to 0.$$

Consequently we have

$$\lim_{n\to+\infty} \{\int_\Gamma \left| \frac{L_{n+\nu}(q_\nu Q_\nu f; z)}{Q_\nu(z)} - Q_\nu(z) f(z) \right|^p |dz|\}^{1/p} = 0. \quad \textbf{(3.4)}$$

Hence by applying Lemma 2 from (3.4) we know

$$\lim_{n\to+\infty} \frac{L_{n+\nu}(q_\nu Q_\nu f; z)}{Q_\nu(z)} = Q_\nu(z) f(z) \quad (3.5)$$

is valid uniformly on any compact inside D.

From that it follows that for sufficient large n the numeretor $P_n(z)$ of $r_{n,\nu}(z)$ doesn't have $\{\beta_j\}$, $1 \leqslant j \leqslant m$, as its zeros. Hence $r_{n,\nu}(z)$ has its poles at zeros of $q_\nu(z)$. From that by using (1.10) we can prove (1.11) in 2° of the Theorem.

At last in virtue of (3.5) and (2.5) we can easily get 4° of the Theorem.

Theorem is proved completely.

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复平面上亚纯函数的有理函数插值逼近

顾 筱 英

（中国地质大学北京研究生院）

摘要 在本文中考虑光滑度为$1+\varepsilon$的Jordan区域D,$\varepsilon>0$,得到了在区域内具有有限个极点的亚纯函数在Fejer插值基点上的有理插值函数的$L^p(\partial D)$空间中的平均逼近阶，$0<p<+\infty$.