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A Remark on the Mahler's and Gelfond's Transference Theorems of Linear Forms*

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Abstract

In the present remark we prove that the Mahler's and Gelfond's transference theorems of linear forms are equivalent essentially, i.e. they are implied each other except varying some unessential constants.

In the diophantine approximation theory the proposition concerning the relationship between two different approximation problems is generally called transference theorem. For the simultaneous homogeneous approximation the most important result is the Mahler's transference theorem of linear forms (see ref. [1]). An useful variant of this theorem is as follow (see ref. [2]):

Theorem A Let $s\geq 2$ be an integer, and $\mathbf{x}=(x_1,\cdots,x_s)$. Let

$$f_k(\mathbf{x})=\sum_{j=1}^{s} f_{kj}x_j \quad (k=1,\cdots,s) \tag{1}$$

and

$$g_k(\mathbf{x})=\sum_{j=1}^{s} g_{kj}x_j \quad (k=1,\cdots,s)$$

be two sets of linear forms in variables $\mathbf{x}$ with real coefficients such that the determinants of coefficients $\det(f_{kj})$, $\det(g_{kj})=d$ all are nonzero and the bilinear form

$$\Phi(\mathbf{x},\mathbf{y})=\sum_{k=1}^{s} f_k(\mathbf{x})g_k(\mathbf{y})=\sum_{i,j=1}^{s} h_{ij}x_iy_j$$

has integer coefficients h_{ij} $(1\leq i,j\leq s)$. Furthermore, Let $\lambda>0$. If the inequalities

$$|f_k(\mathbf{x})|\leq\lambda \quad (k=1,\cdots,s) \tag{3}$$

are soluble with integral $\mathbf{x}\neq\mathbf{0}$, then the inequalities

$$|g_k(\mathbf{y})|\leq(s-1)(|d|\lambda)^{1/(s-1)}, \quad (k=1,\cdots,s) \tag{4}$$

are also soluble with integral $\mathbf{y}\neq\mathbf{0}$.

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In 1960s Gelfond established another transference theorem of linear forms by Siėgel's analytic method (see ref. [3]). We restate it as the following

Theorem B Let $s\geq 2$ be an integer, and $\mathbf{x}=(x_1,\cdots,x_s)$. Let

$$A_k(\mathbf{x})=\sum_{j=1}^{s}a_{kj}x_j \quad , \quad (k=1,\cdots,s) \tag{5}$$

be a set of linear forms in variables $\mathbf{x}$ with real coefficient such that the determinant of coefficients $\det(a_{kj})=\Delta$ is nonzero. Denoting the inverse of the matrix (a_{kj}) by the matrix (b_{kj}), we put the linear forms in variables $\mathbf{x}$.

$$B_k(\mathbf{x})=\sum_{j=1}^{s}b_{jk}x_j \quad , \quad (k=1,\cdots,s). \tag{6}$$

Furthermore, let $t_1,\cdots,t_s,\rho,\tau$ be positive real numbers satisfying

$$0<\rho\leq\tau\leq\frac{1}{\pi}\left(1-\frac{1}{4s+1}\right)^{2s}\sqrt{\frac{6}{4s+1}} \tag{7}$$

and

$$t_1\,t_2\cdots t_s\geq\frac{\rho}{\tau|\Delta|} \tag{8}$$

If the inequalities

$$|A_k(\mathbf{x})|\leq\frac{\rho}{t_k} \quad (k=1,\cdots,s) \tag{9}$$

are soluble with integral $\mathbf{x}\neq\mathbf{0}$, then the inequalities

$$|B_k(\mathbf{y})|\leq t_k, \quad (k=1,\cdots,s) \tag{10}$$

are also soluble with integral $\mathbf{y}\neq\mathbf{0}$.

In the bibliography Theorem A and Theorem B are independent each other. But in the application these two theorems often appear to have the same power. For example, the well-known Khintchine's transference principal is an immediate consequence of Theorem A (see e.g.ref.[2]), but Gelfond was succesful in deducing it from Theorem B (see ref. [3]). Furthermore, by means of either Theorem A or Theorem B ones could generalize the Khinchin's trasference principal (see refs. [4] and [5]). Another such an example can be found in ref. [6]. In the present remark we prove that the Mahler's and Gelfond's transference theorems of linear forms are equivalent essentially, that is the following

Theorem C Theorem A and Theorem B are implied each other except varying the constants on the right sides of the inequalities (4) and (10).

Remark In the application the constants mentioned above are unessential.

Proof First prove that Theorem A implies Theorem B. Using the matrix denotation, let the matrices $A=(a_{kj})$ and $B=(b_{kj})$. Then we have $A'B=I$ by the hypothesis of Theorem B, where I is the unit matrix. It is clear that

$$(A_1(\mathbf{x}),\cdots,A_s(\mathbf{x}))'=A\mathbf{x}', \qquad (B_1(\mathbf{y}),\cdots,B_s(\mathbf{y}))'=B\mathbf{y}',$$

and so the bilinear form

$$\varphi(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{s} A_k(\mathbf{x}) B_k(\mathbf{y})$$

$$= (A_1(\mathbf{x}), \cdots, A_s(\mathbf{x}))(B_1(\mathbf{y}), \cdots, B_s(\mathbf{y}))' = \mathbf{x} A' B \mathbf{y}' = \sum_{k=1}^{s} x_k y_k ,$$

In Theorem A taking

$$f_k = \frac{t_k}{\rho} A_k, \quad g_k = \frac{\rho}{t_k} B_k \quad (k = 1, \cdots, s),$$

and $d = \rho^s/(t_1 \cdots t_s \Delta)$, $\lambda = 1$, then we deduce that the inequalities

$$\left|\frac{\rho}{t_k} B_k(\mathbf{y})\right| \leq (s-1)\rho^{s/(s-1)}(t_1 \cdots t_s |\Delta|)^{-1/(s-1)}$$

are soluble with integral $\mathbf{y} \neq \mathbf{0}$. Thereby, noticing the inequality (8), the inequalities

$$|B_k(\mathbf{y})| \leq (s-1)\tau^{1/(s-1)} t_k, \quad (k = 1, \cdots, s)$$

are soluble with integral $\mathbf{y} \neq \mathbf{0}$. Here the constant on the right side of the inequality is $(s-1)\tau^{1/(s-1)}$, which replaces the corresponding constant 1 in the set of inequalities (10).

Next we prove that Theorem B implies Theorem A.

First we assume that the constant d and the parameter λ satisfy

$$|d|\lambda^s \leq \tau_0^s, \tag{11}$$

where

$$\tau_0 = \frac{1}{\pi}\left(1 - \frac{1}{4s+1}\right)^{2s} \sqrt{\frac{6}{4s+1}} .$$

Denote the matrices

$$F = (f_{kj}), \quad G = (g_{kj}), \quad H = (h_{ij}) .$$

Then the bilinear form

$$\Phi(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{s} f_k(\mathbf{x}) g_k(\mathbf{y}) = \mathbf{x} F' G \mathbf{y} .$$

But it can be represented as $\mathbf{x} H \mathbf{y}'$ by the bypothesis of Theorem A, hence we have

$$H = F' G. \tag{12}$$

Since both detF and detG are all nonzero hence $\det H \neq 0$. Let $\mathbf{w} = (w_1, \cdots, w_s) = \mathbf{x}H$. Then $\mathbf{w}$ is integral when $\mathbf{x}$ is integral, and $\mathbf{w} \neq \mathbf{0}$ when $\mathbf{x} \neq \mathbf{0}$. We define the linear forms in variables $\mathbf{w}$

$$\widetilde{f}_k(\mathbf{w}) = f_k(\mathbf{w}H^{-1}) \quad (k = 1, \cdots, s).$$

From the inequalities (3) we deduce that the inequalities

$$|\widetilde{f}_k(\mathbf{w})| \leq \lambda \quad (k = 1, \cdots, s) \tag{13}$$

are soluble with integral $\mathbf{w} \neq \mathbf{0}$. By the matrix $\widetilde{F}$ denote the matrix of coefficients of the linear forms $\widetilde{f}_k(\mathbf{w})$ $(k = 1, \cdots, s)$. We have

$$(\widetilde{f}_1(\mathbf{w}), \cdots, \widetilde{f}_s(\mathbf{w}))' = \widetilde{F}\mathbf{w}'.$$

On the other hand, we have

$$(f_1(\mathbf{w}H^{-1}), \cdots, f_s(\mathbf{w}H^{-1}))' = F(\mathbf{w}H^{-1})' = FH'^{-1}\mathbf{w}' ,$$

hence $\widetilde{F}=FH'^{-1}$. And so we obtain $\widetilde{F}G'=I$ by (12), i.e. the matrix of coefficients of the linear forms $g_k(\mathbf{y})$ $(k=1,\cdots,s)$ is the transpose of the inverse of the matrix of coefficients of the linear forms $\widetilde{f}_k(\mathbf{w})$ $(k=1,\cdots,s)$.

Now in Theorem B taking the constant $\tau=\tau_0$ and

$$\rho=\tau_0^{-1/(s-1)}(|d|\lambda^s)^{1/(s-1)},$$

and the parameters

$$t_1=t_2=\cdots=t_s=\tau_0^{-1/(s-1)}(|d|\lambda)^{1/(s-1)}, \tag{14}$$

by (11) we have

$$0<\rho\leq\tau_0,\ t_1\ t_2\ \cdots\ t_s=\rho|d|/\tau_0.$$

Furthermore the inequalities (13) can be rewritten as

$$|\widetilde{f}_k(\mathbf{w})|\leq\frac{\rho}{t_k}\quad(k=1,\cdots,s).$$

Thus from Theorem B we deduce that the inequalities

$$|g_k(\mathbf{y})|\leq\tau_0^{-1/(s-1)}(|d|\lambda)^{1/(s-1)}\quad(k=1,\cdots,s) \tag{15}$$

are soluble with integral $\mathbf{y}\neq\mathbf{0}$.

If the inequality (11) does not hold, then $|d|\lambda^s>\tau_0^s$, and so it is easy to see that the parameters $t_1,\cdots,t_s$ in (14) satisfy

$$\begin{aligned}t_1\ t_2\ \cdots\ t_s&=\tau_0^{-s/(s-1)}(|d|\lambda)^{s/(s-1)}\\&=|d|\tau_0^{-s/(s-1)}(|d|\lambda^s)^{1/(s-1)}>|d|\tau_0^{-s/(s-1)}(\tau_0^s)^{1/(s-1)}=|d|.\end{aligned}$$

By the Minkowski's theorem of linear forms (see ref.[2]) the inequalities (15) also have integral solution $\mathbf{y}\neq\mathbf{0}$.

Notice that the constant $\tau_0^{-1/(s-1)}$ in (15) replaces the constant $s-1$ in (4). In paticular, for large s this new constant is smaller than the old one. Q. E. D.

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关于Mahler和Gelfond的线性型转换定理

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摘　要

本文证明了Mahler和Gelfond的两个线性型转换定理本质上是等价的，亦即除去某些非本质性的常数因子改变外，两个定理是互相蕴含的.