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Uniform Approximation by the Combinations of Ismail–May Operators*

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1. Introduction

In 1976, C.P. May [1] introduced the exponential-type operators $L_n(f;x)$. For $f \in C(A,B)$

$$L_n(f;x) = \int_A^B w(n,x,u)f(u)du,$$

where

$$w(n,x,u) \geq 0, \quad \int_A^B w(n,x,u)du = 1. \tag{1}$$

$$\frac{\partial w(n,x,u)}{\partial x} = \frac{n}{\varphi^2(x)} w(n,x,u)(u-x),$$

and $C(A,B)$ denotes the set of continuous and bounded functions defined on (A,B), (A,B) may be $[0,1]$, $[0,\infty]$ or $(-\infty,+\infty)$ etc.

In 1978, E.H. Ismail and C.P. May [2] pointed out that for $\varphi^2(x) = ax^2+bx+c$, there are only six essential exponential-type operators: Gauss-Weierstrass operator $W_n(f;x)$, Szasz-Mirakjan operator $S_n(f;x)$, Bernstein operator $B_n(f;x)$, Baskakov operator $V_n(f;x)$, Post- Widder operator $P_n(f;x)$ and Ismail-May operator $T_n(f;x)$.

In 1988, V. Totik [3] studied the uniform approximation by all of the above operators. Recently, Z. Ditzian and V. Totik [4] investigated the uniform approximation by the combination of exponential-type operators: $B_n(f;x)$, $S_n(f;x)$, $V_n(f;x)$ and $P_n(f;x)$ and give the direct and inverse theorems and the characterization. In [5], we discussed the uniform approximation by the combination of $W_n(f;x)$. Here, we shall study the uniform approximation by the following combination of Ismail-May operator $T_n(f;x)$.

Let $T_n(f;x)$ denote Ismail-May operator:

$$T_n(f;x) = \frac{2^{n-2}n}{\pi(n-1)!}(1+x^2)^{-\frac{n}{2}} \int_{-\infty}^{\infty} e^{nu \arctan x} \left| \Gamma(n/2 + i\frac{nu}{2}) \right|^2 f(u)du, \tag{2}$$

and $T_{n,r}(f;x)$ denote a linear combination of $T_n(f;x)$:

$$T_{n,r}(f;x) = \sum_{i=0}^{r} C_i(n) T_{ni}(f;x). \tag{3}$$

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Where n_i and $C_{i(n)}$ satisfy

$$\begin{cases} \text{(a)} & n = n_0 < n_1 < \cdots < n_r \le k_n, \\ \text{(b)} & \sum_{i=0}^{r} | C_i(n) | < C, \\ \text{(c)} & \sum_{i=0}^{r} C_i(n) = 1, \\ \text{(d)} & \sum_{i=0}^{r} C_i(n) n_i^{-p} = 0, \quad p = 1(1)r. \end{cases} \tag{4}$$

In this paper, we shall prove the following results:

Theorem 1.1 *For $f \in C_{(-\infty,\ \infty)}$, we have*

$$\| T_{n,r}(f) - f \| \le M \left[\omega_\varphi^{2r}(f; n^{-1/2}) + n^{-r} \| f \| \right], \tag{5}$$

$$K_{2r,\varphi}(f; n^{-r}) \le \| T_{k,r}(f) - f \| + M(k/n)^r K_{2r,\varphi}(f, k^{-r}), \tag{6}$$

and for $0 < \alpha < 2r$

$$\| T_{n,r}(f) - f \| = O(n^{-\alpha/2}) \Longleftrightarrow \omega_\varphi^{2r}(f; h) = O(h^\alpha). \tag{7}$$

Where $\| \cdot \| = \sup \| \cdot \|$, M denotes a constant independent of f, n and x, (in what follows, M will always denote different constants).

$$\omega_\varphi^{2r}(f; t) = \sup_{0<h\le t} \| \Delta_\varphi^{2r} f(x) \|,$$

$$\varphi^2(x) = 1 + x^2,$$

and

$$\Delta_h^r f(x) = \sum_{k=0}^{r} (-1)^k f(x + \frac{hr}{2} - kh),$$

$K_{r,\varphi}, (f, t)$ denotes the k–th functional

$$K_{r,\varphi}(f, t_r) = \inf_g \{ \| f - g \| + t^r \| \varphi^r g^{(r)} \| \}, \tag{8}$$

where

$$g \in D = \{ g \mid g \in C_{(-\infty,\infty)}, g^{(r-1)} \in \text{ A.C. loc }, \| \varphi^r g^{(r)} \| < \infty \}.$$

2. Lemmas

To prove Theorem 1.1, we need some Lemmas.

Lemma 2.1 *For $f \in C(R)$,*

$$\| T_{n,r}(f) \| \le M \| f \| . \tag{9}$$

Proof From the definition (3) and $\| T_n(f; x) \| \le 1$, we have

$$\| T_{n,r}(f) \| \le \sum_{i=o}^{r} | C_i(n) | \cdot | T_{hi}(f; x) | \le \sum_{i=o}^{r} | C_i(n) | \| f \| .$$

By (4) , (9) is trivial. □

Lemma 2.2 *For* $f \in C(R)$

$$\| \varphi^{2r} T_n^{(2r)}(f;x) \| \le M n^r \| f \| . \tag{10}$$

Proof Let

$$W(n,x,u) = \frac{2^{n-2} n}{\pi(n-1)!}(1+x^2)^{-n/2} \left| \Gamma(n/2 + i\frac{nu}{2}) \right|^2 e^{nu \arctan x}.$$

From (1), we have

$$\frac{\partial W(n,x,u)}{\partial x} = \varphi^{-2} W(n,x,u)[n(u-x)], \tag{11}$$

by derivating (12) and from (1), we have

$$\frac{\partial^2 W(n,x,u)}{\partial x^2} = (\varphi^{-2})^2 W(n,x,u)\{[n(u-x)]^2 - n(u-x)2\varphi\varphi' - n\varphi^2\}.$$

Since $\varphi^2 = 1 + x^2$, then $2\varphi\varphi' = 2x$, therefore

$$\frac{\partial^2 W(n,x,u)}{\partial x^2} = (\varphi^{-2})^2 W(n,x,u)\{[n(u-x)]^2 - n(u-x)2x - n\varphi^2\}.$$

Generally, from (1) and by simple calculations, we have

$$\begin{aligned} \frac{\partial^{2r} W(n,x,u)}{\partial x^2} &= (\varphi^{-2})^{2r}[n(u-x)]^{2r} W(n,x,u) \\ &\quad + \sum_{i=0}^{2r-1} (\varphi^{-2})^{2r} Q_i(n,x)(u-x)^i W(n,x,u). \end{aligned} \tag{12}$$

Where $Q_i(n,x)$ is a polynomial in n and x with constant coefficients, and satisfies

$$| (\varphi^{-2})^{2r} Q_i(n,x) | \le C(\frac{n}{\varphi^2})^{r+i/2}, \tag{13}$$

where C is a constant.

Therefore

$$\begin{aligned} | \varphi^{2r} T_n^{(2r)}(f;x) | &= \left| \int_{-\infty}^{\infty} \frac{\partial^{2r} W(n,x,u)}{\partial x^{2r}} f(u)\, du \right| \\ &\le \left| \int_{-\infty}^{\infty} \varphi^{-2r} n^{2r} W(n,x,u)(u-x)^{2r} f(u) du \right| \\ &+ \left| \int_{-\infty}^{\infty} \sum_{i=0}^{2r-1} \varphi^{-2r} Q_i(n,x)(u-x)^i W(n,x,u) f(u)\, du \right| \\ &\triangleq I_1 + I_2. \end{aligned}$$

Let $A_i(n,x) = n^i \int_{-\infty}^{\infty} W(n,x,u)(u-x)^i\,du$. By [1], [4] we have

$$A_0 = 1, \quad A_1 = 0, \quad A_{m+1}(u,x) = mn\varphi^2 A_{m-1}(n,x) + \varphi^2 \frac{d}{dx} A_m(n,x),$$
$$A_{2r}(n,x) \le C\varphi^{2r} n^r. \tag{14}$$

So we get

$$| I_1 | \le \| f \| \varphi^{-2r} A_{2r}(n,x) \le M \| f \| n^r. \tag{15}$$

Using (13), (14) and Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} | I_2 | &\le \| f \| \sum_{i=0}^{2r-1} \varphi^{2r} C \left(\frac{n}{\varphi^2}\right)^{r+i/2} \left\{ \int_{-\infty}^{\infty} W(n,x,u)(u-x)^{2i} \right\}^{1/2} \\ &\le \| f \| \sum_{i=0}^{2r-1} \varphi^{2r} C_1 \left(\frac{n}{\varphi^2}\right)^{r+i/2} n^{-i} n^{i/2} \varphi^i \le M \| f \| n^r. \end{aligned} \tag{16}$$

From (15), (16) (10) is trivial. □

Lemma 2.3 *For $f^{(2r-1)} \in$ A.C. loc.,*

$$\| T_{n,r}(f) - f \| \le M n^{-r} (\| \varphi^{2r} f^{(2r)} \| + \| f \|). \tag{17}$$

Proof Expanding $f(u)$ by Taylor's formula:

$$f(u) = \sum_{i=0}^{2r-1} \frac{f^{(i)}(x)}{i!}(u-x)^i + R_{2r}(f,u,x),$$

where $R_{2r}(f,u,x) = \int_x^u \frac{f^{(2r)}(t)}{(2r)!}(u-t)^{(2r-1)}\,dt$, therefore

$$\begin{aligned} | T_{n,r}(f) - f | &\le \left| \sum_{i=1}^{2r-1} \frac{f^{(i)}(x)}{i!} T_{n,r}((u-x)^i; x) \right| \\ &+ T_{n,r}(R_{2r}(f,u,x); x) \triangleq J_1 + J_2. \end{aligned} \tag{18}$$

Using (13), and by a simple calsulation, we have

$$\begin{aligned} | J_2 | &\le M \| \varphi^{2r}(x) f^{(2r)}(x) \| \varphi^{(-2r)}(x) \sum_{j=0}^{r} | C_j(n) | \int_{-\infty}^{\infty} W(n,x,u)(u-x)^{2r}\,du \\ &= M \| \varphi^{2r} f^{(2r)}(x) \| \varphi^{(-2r)}(x) \sum_{j=0}^{r} | C_j(n) | n^{-2r} A_{2r}(n,x) \\ &\le M \| \varphi^{2r} f^{(2r)}(x) \| \varphi^{(-2r)} n^{-r} \varphi^{2r} = M \| \varphi^{2r}(x) f^{(2r)}(x) \| n^{-r}, \end{aligned} \tag{19}$$

by [1], $A_i(n,x)$ is a polynomial in n of degree $[i/2]$ and in x of degree i, therefore,

$$\begin{aligned}
|J_1| &= \left|\sum_{i=o}^{2r-1}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{r}C_j(n)\int_{-\infty}^{\infty}W(n,x,u)(u-x)^i\,du\right| \\
&= \left|\sum_{i=o}^{2r-1}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{r}C_j(n)C_i(x)n^{-i+[i/2]}\right| \\
&= \left|\sum_{i=o}^{2r-1}\frac{f^{(i)}(x)}{i!}C_i(x)\sum_{j=0}^{r}C_j(n)n^{-i+[i/2]}\right|.
\end{aligned}$$

Using (4) (a), for $i=1,2,\cdots,2r-1$, we have $\sum_{j=0}^{r}C_j(n)n^{-i+[i/2]}=0$, then

$$J_1 = 0. \tag{20}$$

From (18)–(20), we can obtain

$$\| T_{n,r}(f)-f \| \le Mn^{-r}\| \varphi^{2r}f^{(2r)} \| \le Mn^{-r}\left(\| \varphi^{2r}f^{(2r)}\| + \| f \|\right).$$

Lemma 2.4 *For $f^{(2r-1)}\in A.C.\ loc.$,*

$$\| \varphi^{2r}T_n^{(2r)}(f) \| \le M \| \varphi^{2r}f^{(2r)} \|. \tag{21}$$

Proof Using Taylor's expansion, we can get

$$\begin{aligned}
|\varphi^{2r}T_n^{(2r)}(f)| &\le \left|\varphi^{2r}(x)\int_{-\infty}^{\infty}\sum_{i=0}^{2r-1}\frac{\partial^{2r}}{\partial x^{2r}}W(n,x,u)\frac{f^{(i)}(x)}{i!}(u-x)^i\,du\right| \\
&+ \| \varphi^{2r}f^{(2r)} \| \left|\int_{-\infty}^{\infty}\left|\frac{\partial^{2r}}{\partial x^{2r}}\right|W(n,x,u)(u-x)^{2r}\,du\right| \\
&\triangleq K_1+K_2.
\end{aligned} \tag{22}$$

By [1], we have $K_1=0$. Using (13)

$$\begin{aligned}
|K_2| &\le M\| \varphi^{2r}f^{(2r)} \| \left|\int_{-\infty}^{\infty}\left|\frac{\partial^{2r}}{\partial x^{2r}}\right|W(n,x,u)(u-x)^{2r}\,du\right| \\
&\le M\| \varphi^{2r}f^{(2r)} \| \left\{\left|\int_{-\infty}^{\infty}(\varphi^{-2})^{2r}n^{2r}(u-x)^{4r}W(n,x,u)\,du\right|\right. \\
&+ \left.\left|\int_{-\infty}^{\infty}\sum_{i=0}^{2r-1}\left|(\varphi^{-2})^{2r}Q_i(n,x)W(n,x,u)(u-x)^{2r+i}\right|du\right|\right\}.
\end{aligned}$$

Since

$$\begin{aligned}
\int_{-\infty}^{\infty}(\varphi^{-2})^{2r}n^{2r}(u-x)^{4r}W(n,x,u)du &= \varphi^{-4r}n^{-2r}A_{4r}(n,x) \\
&\le M\varphi^{-4r}n^{-2r}n^{2r}\varphi^{4r}=M.
\end{aligned}$$

Using (13) and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} \sum_{i=0}^{2r-1} \mid (\varphi^{-2})^{2r} Q_i(n,x) W(n,x,u)(u-x)^{2r+i} \mid du \right| \\
\leq\ & M \sum_{i=0}^{2r-1} \left(\int_{-\infty}^{\infty} W(n,x,u)(u-x)^{4r+2i} du \right)^{1/2} (n/\varphi^2)^{r+i/2} \\
\leq\ & M \sum_{i=0}^{2r-1} (n/\varphi^2)^{r+i/2} n^{-(2r+i)} n^{(2r+i)/2} \varphi^{2r+i} \leq M.
\end{aligned}$$

Therefore $| K_2 | \leq M \| \varphi^{2r} f^{(2r)} \|$ and $\| \varphi^{2r} T_n^{(2r)}(f) \| \leq | K_1 | + | K_2 | \leq M \| \varphi^{2r} f^{(2r)} \|$.

3. The proof of Theorem 1.1

Let D be a weighted Sobolev space

$$D = \{ g \ \mid g \in C_{(R)}, g^{(2r-1)} \in \text{ A.C. loc } , | \varphi^{2r} g^{(2r)} | < \infty \}.$$

for a $g \in D$. Then we have

$$\| T_{n,r}(f) - f \| \leq \| T_{n,r}((f-g);x) \| + \| f - g \| + \| T_{n,r}(g;x) - g \| .$$

By Lemma 2.1, we have

$$\| T_{n,r}((f-g);x) \| \leq \| f - g \| .$$

By Lemma 2.3, we have

$$\| T_{n,r}(g) - g \| \leq M n^{-r} \left(\| \varphi^{2r} g^{(2r)} \| + \| g \| \right) .$$

Therefore

$$\| T_{n,r}(f) - f \| \leq 2 \| f - g \| + M n^{-r} \| \varphi^{2r} g^{(2r)} \| + M n^{-r} \| f \| .$$

Take in f for g on two sides, we have

$$\| T_{n,f}(f) - f \| \leq M K_{2r\varphi}(f; n^{-r}) + M n^{-r} \| f \| .$$

By [4]

$$K_{2r,\varphi}(f; n^{-r}) \sim \omega_{\varphi}^{2r}(f; n^{-1/2}), \tag{23}$$

then (5) follows.

Write $f = f - T_{k,r}(f) + T_{k,r}(f)$, we have

$$K_{2r,\varphi}(f; n^{-1/2}) \leq \| f - T_{k,r}(f) \| + n^{-r} \| \varphi^{2r} T_{k,r}^{(2r)}(f) \| . \tag{24}$$

Since $\| \varphi^{2r} T_{k,r}^{(2r)}(f) \| \leq \| \varphi^{2r} T_{k,r}^{(2r)}(f-g) \| + \| \varphi^{2r} T_{k,r}^{(2r)}(g) \|$, where $g \in D$. By Lemma 2.2

$$\| \varphi^{2r} T_{k,r}^{(2r)}(f-g) \| \leq M K^r \| f - g \| .$$

By Lemma 2.4,

$$\| \varphi^{2r} T_{k,r}^{(2r)}(g) \| \leq M \| \varphi^{2r} g^{(2r)} \| . \tag{25}$$

Therefore, we have

$$\begin{aligned} n^{-r} \| \varphi^{2r} T_{k,r}^{(2r)}(f) \| &\leq M(k/n)^r \| f-g \| + Mn^{-r} \| \varphi^{2r} g^{(2r)} \| \\ &\leq M(k/n)^r \left(\| f-g \| + K^{-r} \| \varphi^{2r} g^{(2r)} \| \right). \end{aligned}$$

Taking inf for g on two sides, we obtain

$$n^{-r} \| \varphi^{2r} T_{k,r}^{(2r)}(f) \| \leq M(k/n)^r K_{2r,\varphi}(f; K^{-r}),$$

from (24), we have

$$K_{2r,\varphi}(f, n^{-1/2}) \leq \| f - T_{k,r}(f) \| + M(k/n)^r K_{2r,\varphi}(f; K^{-r}).$$

This is (6).

Since

$$\| T_{k,r}(f) - f \| \leq \| T_{n,r}(f) - f \| + \| T_{n,r}(f) - T_{k,r}(f) \|, \tag{26}$$

using (5), we have

$$\| T_{n,r}(f) - T_{k,r}(f) \| \leq M\omega_\varphi^{2r} \left(T_{k,r}(f); n^{-1/2} \right) + M \| T_{k,r}(f) \| n^{-r},$$

by [4], we have $\omega_\varphi^{2r}(T_{k,r}(f); n^{-1/2}) \leq Mn^{-r}$. by (10), we have $\| T_{k,r}(f) \| \leq \| f \|$. Therefore

$$\| T_{n,r}(f) - T_{k,r}(f) \| \leq Mn^{-r} \| f \| . \tag{27}$$

To replace (27) to (26), we obtain

$$\| T_{k,r}(f) - f \| \leq \| T_{n,r}(f) - f \| + M \| f \| n^{-r}.$$

Using (6), we have

$$K_{2r,\varphi}(f; n^{-r}) \leq \| Tn, r(f) - f \| + M \| f \| n^{-r} + M(k/n)^r K_{2r,\varphi}(f, k^{-r}).$$

By the Berens-Lorentz Lemma [4] and (5), for $0 < \alpha < 2r$, we have

$$\begin{aligned} K_{2r,\varphi}(f; n^{-r}) &\leq \| T_{n,r}(f) - f \| + M \| f \| n^{-r} \\ &\leq M\omega_\varphi^{2r}(f; n^{-\frac{1}{2}}) + M \| f \| n^{-r}, \end{aligned}$$

from (23) , then

$$\| T_{n,r}(f) - f \| = O(n^{-\alpha/2}) \Longleftrightarrow \omega_\varphi^{2r}(f; h) = O(h^\alpha).$$

This completes the proof of Theorem 1.1.

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关于 Ismail－May 算子线性组合的一至逼近

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摘　　要

设 $T_n(f;x)$ 表示如下的 Ismail－May 算子

$$T_n(f;x)=\frac{2^{n-2}n}{\pi(n-1)!}(1+x)^{-\frac{n}{2}}\int_{-\infty}^{\infty}e^{nu\operatorname{arctg}x}\left|\Gamma\left(\frac{n}{2}+\frac{inu}{2}\right)\right|^2 f(u)du.$$

它是一类指数型算子，其逼近阶不会超过 $O(\frac{1}{n})$. 为提高它的逼近阶，在本文中，我们给出子 $T_n(f;x)$ 的一类线性组合

$$T_{n,r}(f;x)=\sum_{i=0}^{r}C_i(n)T_{n_i}(f;x).$$

借助于 K－泛函 $K_{2r,\phi}(f;n^{-r})$ 与光滑模 $w_\phi^{2r}(f,h)$ 之间的等价性，我们讨论了它在一致逼近意义下的正定理，逆定理和特征刻划问题，即有

定理 1.1　如 $f\in C_{B(-\infty,\infty)}$，则

$$\|T_{n,r}(f)-f\|\leqslant M(w_\varphi^{2r}(f;n^{\frac{1}{2}})+\|f\|n^{-r}),$$

$$K_{2r,\varphi}(f;n^{-r})\leqslant\|T_{k,r}(f)-f\|+M(\frac{K}{n})^r K_{2r,\varphi}(f,K^{-r}).$$

当　$0<\alpha<2r$　$(r>1)$时

$$\|T_{n,r}(f)-f\|=O(n^{-\frac{\alpha}{2}})\Leftrightarrow w_\varphi^{2r}(f,h)=O(h^\alpha).$$