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On the Convergence of Generalized Dirichlet Series with Complex Exponents*

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There are two kinds of formulas which are often applied to determine the convergent abscissa of the ordinary Dirichlet series. One is Valiron's and the other is Knopp's. Yu Jiarong has studied the convergence of the generalized Dirichlet series with complex exponents in [1] and given a convergent formula which is similar to Valiron's. In this paper we will extend Knopp's formula on the convergent abscissa of the ordinary Dirichlet series to generalized Dirichlet series with complex exponents.

Suppose that the generalized Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \tag{1}$$

where $\lambda_n = \omega_n e^{i\tau_n}, \omega_n > 0, 0 \le \tau_n < 2\pi; s = \sigma + it, \sigma, t$ being real variables; and $\{a_n\}$ is a sequence of complex numbers.

In this paper we suppose that $\{\lambda_n\}$ satisfies the condition C:

1. $|\lambda_n| \le |\lambda_{n+1}|$,
2. $\lim\limits_{n\to} |\lambda_n| = +\infty$,
3. $\lambda_n \ne \lambda_m$ if $n \ne m$.

And

$$\begin{aligned} H &= \overline{\lim_{n\to\infty}}[(P_m - P_{m-1})/m] < +\infty, \\ -\infty &< l = \overline{\lim_{n\to\infty}}(\ln A_m)/m < +\infty, \end{aligned}$$

where

$$\begin{aligned} A_m &= \max_{P_{m-1}+1 \le k \le m} \{|a_{P_{m-1}+1} + a_{P_{m-1}+2} + \cdots + a_k|\}, \\ |\lambda_{P_m}| &\le m < |\lambda_{P_m+1}|. \end{aligned}$$

On $s = \sigma + it$ plane, make a straight line $L_\theta(-\pi/2 < \theta \le \pi/2)$, which passes through the origin and its oblique angle is θ. Then the point $s = \sigma + it$ on the plane may be expressed by polar coordinates $(r, \theta), -\infty < r < +\infty$, and $\sigma = r\cos\theta, t = r\sin\theta$.

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Under these conditions, we may consider convergence of the series (1) on the L_θ and obtain the following conclusions:

Theorem 1 *Suppose that* $s = re^{i\theta} \in L_\theta$. *If* $r_1^-(\theta) < r < r_1^+(\theta)$, *the series (1) absolutely converges at the point s. If* $r > r_2^+(\theta)$ *or* $r < r_2^-(\theta)$, *the series (1) diverges at the point s, where*

$$r_1^+(\theta) = l\overline{\lim_{j\to\infty}}[1/\cos(\tau_{kj}+\theta)],$$

$$r_1^-(\theta) = l\underline{\lim}_{j\to\infty}\ [1/\cos(\tau_{mj}+\theta)],$$

$$r_2^-(\theta) = \overline{\lim_{j\to\infty}}\frac{\ln A_{m_{mj}} - \ln(P_{m_{mj}} - P_{m_{mj}-1})}{\omega_{m^*_{mj}}\cos(\tau_{m^*_{mj}}+\theta)},$$

$$r_2^+(\theta) = \underline{\lim}_{j\to\infty}\ \frac{\ln A_{m_{kj}} - \ln(P_{m_{kj}} - P_{m_{kj}-1})}{\omega_{m^*_{kj}}\cos(\tau_{m^*_{kj}}+\theta)}.$$

Where $\{kj\},\{mj\}$ are sequences that are chosen to be suitable to the condition $\cos(\tau_{kj} + \theta) \le 0, \cos(\tau_{mj}+\theta) > 0$ $(j = 1,2,3,\cdots)$ in the natural numbers according to the natural order.

$\{m^*_{mj}\},\{m^*_{kj}\}$ are subsequences of the sequence $\{m^*\}$ satisfying $\cos(\tau_{m^*_{kj}}+\theta) \le 0$, $\cos(\tau_{m^*_{mj}} + \theta) > 0$, but $\{m^*\}$ is the subsequence of the natural number sequence and $P_{m-1}+1 \le m^* \le P_m, |a_{m^*}| = \max\{|a_{P_{m-1}+1}|, |a_{P_{m-1}+2}|,\cdots,|a_{P_m}|\}$; $\{m_{kj}\},\{m_{mj}\}$ are the subsequences of the natural number sequence $\{m\}$ satisfying $\cos(\tau_{m^*_{kj}}+\theta) \le 0, \cos(\tau_{m^*_{mj}} + \theta) > 0$.

In the expression of the $r_1^+(\theta)$ and $r_2^+(\theta)$, if $\cos(\tau_{kj}+\theta) = 0$, then $1/\cos(\tau_k+\theta)$ is taken as $-\infty$ and if $\cos(\tau_{m^*_{kj}}+\theta) = 0$, $[\ln A_{m_{kj}} - \ln(P_{m_{kj}} - P_{m_{kj}-1})]/[\omega_{m^*_{kj}}\cos(\tau_{m^*_{kj}}+\theta)]$ is taken as $+\infty$ or $-\infty$ according as $\ln A_{m_{kj}} - \ln(P_{m_{kj}} - P_{m_{kj}-1})$ being negative or nonnegative.

Proof We only prove the theorem in the case where $r_1^+(\theta), r_1^-(\theta), r_2^+(\theta), r_2^-(\theta)$ are all finite numbers, the others may be verified by similar means.

(1) If $r_1^-(\theta) < r < r_1^+(\theta)$, evidently $l < 0$. There must exist an positive number $\delta_0 < |l|$ such that for $0 < \delta < \delta_0$,

$$(l+\delta)\underline{\lim}_{j\to\infty}\ 1/\cos(\tau_{mj}+\theta) < r < (l+\delta)\overline{\lim_{j\to\infty}}1/\cos(\tau_{kj}+\theta).$$

Then there exists an integer $N_0 > 0$ such that for $K_j > N_0, m_j > N_0$,

$$1/\cos(\tau_{mj}+\theta) > r/(l+\delta) > 1/\cos(\tau_{kj}+\theta).$$

Hence when $0 < \delta < \delta_0$ and $n > N_0$,

$$r\cos(\tau_n+\theta) > l+\delta.$$

On the other hand for the aforementioned δ, there exists an integer $N_1 > 0$ such that for any $m > N_1, A_m < e^{m(l+\delta/2)}$. However, when $P_{m-1}+1 \le n \le P_m$, we have

$$|a_n| \le 2A_m < 2e^{m(l+\delta/2)}.$$

Consequently when $n > N_0, P_{m-1}+1 \le n \le P_m$ and $m > N_1, |a_n e^{-\lambda_n s}| \le 2e^{-m\delta/2}$.

We can easily prove that there exists an integer $N_2 > 0$ such that for any $m > N_2$

$$e^{-m\delta/4} < \frac{1}{m}, \quad (P_m - P_{m-1})/m < c,$$

here c is a positive constant.

Hence when $P_{m-1} > N_0, m > \max\{N_1, N_2\}$

$$\sum_{n=P_{m-1}+1}^{P_m} |a_n e^{-\lambda_n s}| \le 2(P_m - P_{m-1})e^{-m\delta/2} < c_1 e^{-m\delta/4},$$

where c_1 is a positive constant.

Because the series $\sum_{m=1}^{\infty} c_1 e^{-m\delta/4}$ converges, this proves that the series (1) absolutely converges.

(2) If $s = re^{i\theta} \in L_\theta, r > r_2^+(\theta)$ or $r < r_2^-(\theta)$, then there exists infinite m_{kj} or m_{mj} satisfying

$$\frac{\ln A_{m_{kj}} - \ln(P_{m_{kj}} - P_{m_{kj}-1})}{\omega_{m_{kj}^*}\cos(\tau_{m_{kj}^*} + \theta)} < r$$

or

$$\frac{\ln A_{m_{mj}} - \ln(P_{m_{mj}} - P_{m_{mj}-1})}{\omega_{m_{mj}^*}\cos(\tau_{m_{mj}^*} + \theta)} > r.$$

Evidently there exists infinite m such that

$$r\omega_{m^*}\cos(\tau_{m^*} + \theta) \le \ln A_m - \ln(P_m - P_{m-1}) \le \ln|a_{m^*}|.$$

Hence there exists infinite m^* such that

$$|a_{m^*}e^{-\lambda_{m^*}s}| = e^{-r\omega_{m^*}\cos(\tau_{m^*}+\theta)+\ln|a_{m^*}|} \ge e^0 = 1.$$

So the series (1) diverges at the point s.

The proof is completed.

Theorem 2 *Given* $s = re^{i\theta} \in L_\theta$.

(1) If $\cos(\tau_n + \theta) > 0$ *for sufficiently large* n, *then for any* $r > r_*^+(\theta)$, *the series (1) absolutely converges and for any* $r < r_*^-(\theta)$ *the series (1) diverges.*

(2) If $\cos(\tau_n + \theta) \le 0$ *for sufficiently large* n, *then for any* $r > r_*^-(\theta)$, *the series (1) absolutely converges and for any* $r < r_*^+(\theta)$, *the series (1) diverges, where*

$$r_*^+(\theta) = \overline{\lim_{n\to\infty}}\, l/\cos(\tau_n + \theta), \quad r_*^-(\theta) = \underline{\lim_{n\to\infty}}\, l/\cos(\tau_n + \theta).$$

Proof We only prove in the case that $r_*^+(\theta), r_*^-(\theta)$ are all finite numbers, the others can be completed in the same way.

(i) When $r > r_*^+(\theta)$ and $\cos(\tau_n + \theta) > 0$ for sufficiently large n, the proof that the series (1) absolutely converges is similar to the proved Theorem 1.

(ii) Let $r < r_*^-(\theta)$ and $\cos(\tau_n+\theta) > 0$ for sufficiently large n, we will prove that the series (1) diverges at the point $s = re^{i\theta}$ by reduction to absurdity.

Suppose that there exists a $s_0 = r_0 e^{i\theta} \in L_\theta$, such that $r_0 < r_*^-(\theta)$ and the series (1) converges at the point s_0.

By $r_0 < \underline{\lim}_{n\to\infty} l/\cos(\tau_n+\theta)$, there exists a $\delta > 0$ such that $r_0 < \underline{\lim}_{n\to\infty} (l-\delta)/\cos(\tau_n+\theta)$, then there exists an integer $N_1 > 0$, such that for $n > N_1, r_0\cos(\tau_n+\theta) < l-\delta$ and $\cos(\tau_n+\theta) > 0$.

When $P_{m-1}+1 \le n \le P_m, m-1 < \omega_n \le m$, hence

$$\begin{aligned} m(l-\delta) &> \omega_n r_0 \cos(\tau_n+\theta), \quad (r_0<0), \\ (m-1)(l-\delta) &> \omega_n r_0 \cos(\tau_n+\theta), \quad (r_0<0). \end{aligned}$$

We can easily prove that for sufficiently large m and n

$$\begin{aligned} (P_m - P_{m-1})/m &< c_1, \quad e^{-m\delta/2} < 1/m, \\ |e^{\lambda_n s_0}| = e^{r_0\omega_n\cos(\tau_n+\theta)} &< c_2 e^{m(l-\delta)}, \\ \Big|\sum_{j=P_{m-1}+1}^{n} a_j e^{-\lambda_j s_0}\Big| &< c_3. \end{aligned}$$

By Abel transform,

$$\begin{aligned} \Big|\sum_{n=P_{m-1}+1}^{k} a_n\Big| &= \Big|\sum_{n=P_{m-1}+1}^{k} a_n e^{-\lambda_n s_0} e^{\lambda_n s_0}\Big| \\ &= \big|(e^{\lambda_{P_{m-1}+1}s_0} - e^{\lambda_{P_{m-1}+2}s_0}) a_{P_{m-1}+1} e^{-\lambda_{P_{m-1}+1}s_0} \\ &\quad + (e^{\lambda_{P_{m-1}+2}s_0} - e^{\lambda_{P_{m-1}+3}s_0})(a_{P_{m-1}+1} e^{-\lambda_{P_{m-1}+1}s_0} \\ &\quad + a_{P_{m-1}+2} e^{-\lambda_{P_{m-1}+2}s_0}) \\ &\quad + \cdots + e^{\lambda_k s_0} \sum_{n=P_{m-1}+1}^{k} a_n e^{-\lambda_n s_0}\big| \\ &< c e^{m(l-\delta/2)}, \end{aligned}$$

where c_1, c_2, c_3, c are all positive constants. Hence we have the contradictory

$$l = \overline{\lim_{m\to\infty}}(A_m/m) \le l - \delta/2.$$

It's proved that the series (1) diverges at the point s_0.

The proof of the conclusion (2) is similar to (1).

When the limit $\lim_{n\to\infty} \tau_n$ exists, we can determine a point $s_0 \in L_\theta$ which cuts L_θ into two half straight lines. The series (1) absolutely converges on one half straight line and it diverges on another. Particularly when $\lim_{n\to\infty} \tau_n = 0$, the series (1) absolutely converges on the half plane and it diverges on the other.

We establish two theorems similar to Abel's theorem and Dirichlet's theorem which determine the convergence of the real number terms series.

Theorem 3 *Suppose that $Im\lambda_n$ is bounded and the series (1) absolutely converges at the point $s_0 = \sigma_0 + it_0$, then for any real number t the series (1) absolutely converges at the point $s = \sigma_0 + it$.*

Proof For arbitrary fixed $s = \sigma_0 + it$, $(-\infty < t < +\infty)$,

$$\begin{aligned} |a_n e^{-\lambda_n s}| &= |a_n e^{-\lambda_n s_0}||e^{\lambda_n (s_0 - s)}| \\ &= |a_n e^{-\lambda_n s_0}|e^{-\omega_n (t_0 - t)\sin\tau_n}. \end{aligned}$$

By $\omega_n \sin\tau_n$ is bounded, we know $e^{-\omega_n(t_0-t)\sin\tau_n}$ is bounded. Evidently the series (1) absolutely converges at the point s.

The proof is completed.

Teheorem 4 *Suppose that the series (1) satisfies the following conditions:*

1. *$H < +\infty$,*
2. *for any integer $k > 0$, $\sum_{n=1}^{k} a_n e^{-\lambda_n s}$ is bounded at the point s_0. If the point s on the plane contents*

$$\overline{\lim_{n\to\infty}} \cos[\tau_n + arg(s_0 - s)] < 0,$$

then the series (1) absolutely converges at the point s.

Proof Suppose that

$$\overline{\lim_{n\to\infty}} \cos[\tau_n + \arg(s_0 - s)] = A < 0,$$

then for sufficiently large n we have

$$\cos[\tau_n + \arg(s_0 - s)] < A/2 < 0.$$

By $\sum_{n=1}^{k} a_n e^{-\lambda_n s_0}$ is bounded, then $|a_n e^{-\lambda_n s_0}|$ is bounded.

Hence for sufficiently large n and s above,

$$\begin{aligned} \sum_{n=P_{m-1}+1}^{P_m} |a_n e^{-\lambda_n s}| &= \sum_{n=P_{m-1}+1}^{P_m} |a_n e^{-\lambda_n s_0}|e^{\omega_n |s_0 - s|\cos[\tau_n + \arg(s_0 - s)]} \\ &\le M(P_m - P_{m-1})e^{m|s_0 - s|A/2} \\ &\le M_1 e^{m|s_0 - s|A/4}, \end{aligned}$$

where M, M_1 are positive constants.

We can easily prove that the series (1) absolutely converges at the point s.

The proof is completed.

References

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广义 Dirichlet 级数的收敛性

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摘 要

本文把决定 Dirichlet 级数收敛横坐标的 Kojima－Knopp 公式推广到复指数 Dirichlet 级数情形.