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The Ring $Z[X]$ is not Chinese*

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Abstract A commutative ring R is said to be Chinese if, given $a, b \in R$ and ideals A, B of R such that $a \equiv b(A+B)$, there exists $c \in R$ such that $c \equiv a(A)$ and $c \equiv b(B)$. Chinese rings were investigated by K.E.Aubert and I.Beck in 1982. However, in their paper, they said that they were unable to settle the case whether the ring $Z[X]$ is Chinese or not. In this paper, we provide a short proof to show that the ring $Z[X]$ is not Chinese. The techinque we used here is different from Aubert and Beck. Moreover, we show that for any algebraic numbers $\alpha_1, \cdots, \alpha_n$, the ring $Z[\alpha_1, \cdots, \alpha_n]$ is Chinese for $n \geq 1$.

Let R be a commutative ring with identity. Any two elements b, c in R are said to be canonically congruent modulo an ideal A of R if $(b, A) = (c, A)$. In other words, b, c are canonically congruent if b and c give rise to residue classes $\bar{b}, \bar{c}$ which generate the same principal ideal $(\bar{b}) = (\bar{c})$ in the quotient ring $\overline{R} = R/A$. We shall denote the canonical congruence of b and c modulo A by $b \equiv c(A)$. Clearly, canonical congruence is a multiplicative congruence which is coarser than the classical congruence modulo the same ideal A in R.

A ring R is said to be Chinese if, given elements $a, b \in R$ and ideals A, B of R such that $a \equiv b(A+B)$ there exists an element $c \in R$ such that $c \equiv a(A)$ and $c \equiv b(B)$. Properties and characterizations of Chinese rings were obtained by K.E.Aubert and I.Beck in 1982[1]. In their paper, they pointed out that polynomial rings with two variables over the ring of integers is not Chinese (see [1], theorem 2). Of course, one would naturally ask whether the ring $Z[X]$ is Chinese or not. Aubert and Beck said that they were unable to settle this case, and so far there is no answer to this problem appeared in the literature.

In this note, we shall demonstrate that $Z[X]$ is also not Chinese, and consequently, $Z[X_1, X_2, \cdots, X_n]$ is not Chinese for $n \geq 1$. Our method used here is quite different from Aubert and Beck [1]. In addition, we also show that for any algebraic numbers $\alpha_1, \cdots, \alpha_n$, the ring $Z[\alpha_1, \cdots, \alpha_n]$ is Chinese for $n \geq 1$. Thus, the result obtained by Aubert and Beck is now strengthened. For definitions and notation, the reader is referred to [1] if necessary.

Theorem 1 *$Z[X]$ is not Chinese.*

proof Consider the elements $X+5$ and $X+10$ in $Z[X]$ .It is easy to check that

$$(X+5, X^2+75, X) = (X+10, X^2+75, X) = (X, 5).$$

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In terms of canonical congruence, the above equality means that

$$X+5 \equiv X+10(X, X^2+75).$$

In other words, $X+5$ and $X+10$ are canonically congruent modulo the ideal (X, X^2+75).

Suppose that $Z[X]$ is Chinese. Then there is a polynomial $z(X) \in Z[X]$ such that

$$z(X) \equiv X+5(X)$$

and

$$z(X) \equiv X+10(X^2+75).$$

Hence we have

$$(z(X), X) = (X+5, X) \tag{1}$$

and

$$(z(X), X^2+75) = (X+10, X^2+75). \tag{2}$$

Putting $X=0$ into (1), we have $(z(0)) = (5)$ in Z and consequently $z(0) = \pm 5$. Thus

$$z(X) = X\omega(X) \pm 5 \tag{3}$$

for some $\omega(X) \in Z[X]$. By (2), we have

$$X+10 = \alpha(X)z(X) + \beta(X)(X^2+75)$$

for some $\alpha(X), \beta(X) \in Z[X]$. Putting $X = \sqrt{-75}$ into the above equation and using (3), we have

$$\begin{aligned}\sqrt{-75}+10 &= \alpha(\sqrt{-75})z(\sqrt{-75}) + \beta(\sqrt{-75}) \cdot 0 \\ &= \alpha(\sqrt{-75})(\sqrt{-75}\omega(\sqrt{-75}) \pm 5).\end{aligned}$$

Simplifying the above equation, we obtain

$$i\sqrt{3}+2 = (c\sqrt{-75}+d)(i\sqrt{3}(a-\sqrt{-75}+b) \pm 1). \tag{4}$$

where $a\sqrt{-75}+b = \omega(\sqrt{-75})$ and $c\sqrt{-75}+d = \alpha(\sqrt{-75})$ for some $a, b, c, d \in Z$. By taking the norms on both sides of (4), we obtain

$$7 = (75c^2+d^2)[3b^2+(15a \pm 1)^2].$$

Since 7 is a prime number, so $(75c^2+d^2) = 1$ or 7. Because c, d are both non-zero integers, so $(75c^2+d^2) \neq 7$. However, if $(75c^2+d^2) = 1$, then $3b^2+(15a \pm 1)^2 = 7$. This is also impossible for any $a, b \in Z$. Thus, $Z[X]$ is not Chinese.

As any quotient ring of a Chinese ring must be Chinese [1], the following corollary is immediate.

Corollary 1.1 *The ring $Z[X_1, \cdots, X_n]$ is not Chinese for $n \geq 1$.*

Remark: The above statement says that any polynomial rings over the ring of integers is not Chinese. Thus, the question raised by Aubert and Beck is now settled. In fact, the theorem obtained by Aubert and Beck (See Theorem in [1]) is included in our Corollary 1.1, but the method of proof is entirely different.

Now, it is known that the ring Z is Chinese but the ring $Z[X]$ is not. It is natural to ask whether the rings sitting in between Z and $Z[X_1, \cdots, X_n]$ are Chinese or not? The following theorem gives an interesting answer.

Theorem 2 *For any algebraic numbers $\alpha_1, \cdots, \alpha_n$, the ring $Z[\alpha_1, \cdots, \alpha_n]$ is Chinese for $n \geq 1$.*

Proof In order to prove theorem 2, we first show that an integral domain R is Chinese if and only if every proper quotient ring of R is Chinese. We only need to prove the sufficiency part as the necessity part had already been proved by Aubert and Beck in [1]. Let $x, y \in R$ and A, B be ideals of R. Suppose that $x \equiv y(A + B)$. Obviously, we may assume that $A \neq (0)$ and $B \neq (0)$. Then $I = A \cap B \neq (0)$ for R is an integral domain. Denote the canonical homomorphic image of $S \subseteq R$ in R/I by $\overline{S}$. Hence $\overline{x} \equiv \overline{y}(\overline{A} + \overline{B})$. Since $\overline{R}$ is assumed to be Chinese, there exists $z \in R$ satisfying the following two equations:

$$(\overline{z}) + \overline{A} = (\overline{x}) + \overline{A},$$

and

$$(\overline{z}) + \overline{B} = (\overline{y}) + \overline{B}.$$

Since $A, B \supseteq I$, the above two equations simply imply that

$$(z) + A = (x) + A,$$

and

$$(z) + B = (y) + B.$$

In order words, $z \equiv x(A)$ and $z \equiv y(B)$. This proves that R is Chinese.

Now we remark that every proper homomorphic image of a Noetherian domain R with Krull dimension one is Chinese. Since for any non-zero proper ideal of R, R/I is an artinian ring (of. [2]). Invoking a result in [1], all the artinian rings are Chinese, thus R/I is Chinese. This proves that every proper homomrophic image of R is Chinese.

Clearly the ring $Z[\alpha_1, \cdots, \alpha_n]$, where $\alpha_1, \cdots, \alpha_n$ $(n \geq 1)$ are algebraic numbers, is a Noetherian domain but not a field. Moreover, the dimension of Z is one and

$$0 < \dim(Z[\alpha_1, \cdots, \alpha_n]) \leq 1.$$

This implies that $\dim(Z[\alpha_1, \cdots, \alpha_n]) = 1$. (for example, see [3]) Thus, using the result just obtained in the last paragraph, we know that every proper homomorphic image of $Z[\alpha_1, \cdots, \alpha_n]$ is Chinese. Hence, by the earlier argument, the ring $Z[\alpha_1, \cdots, \alpha_n]$ is Chinese as well. The proof is now completed.

Remark In fact, theorem 2 still holds if our base ring Z is to be replaced by any arbitrary Noetherian domain with Krull dimension one. The proof is exactly the same as above.

References

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环 Z[X] 不是中华环

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摘　要

设 R 是具有单位元 1 的交换环；A 是 R 中的理想而 a，b 则是 R 中的任意元. 定义 $a\equiv b(A)$ 若 $Ra+A=Rb+A$. 称环 R 是中华环若 $a\equiv b(A+B)$，则存在 $c\in R$ 使 $c\equiv a(A)$ 及 $c\equiv b(B)$. 环是中华环的充要条件是由 K. Aubert 与 A. Beck 二人于 1980 年找出的. 显然，整数环 Z 必是中华环. Aubert 与 Beck 二人亦证明了 Z[x,y]不是中华环. 但他们二人无法证明 Z[X]是否中华环. 本文用不同的手法处理，证明了 Z[X]不可能是中华环. 同时，我们进一步证明，对任意代数数 α，环 Z[α]均是中华环. 因此，Aubert 与 Beck 在 1980 年所提出的问题，在本文中得到圆满的解答.