Theorem 14.3.1]). Again from lemma 1.(2) we see that $|H| > n$. It is contrary to G being an OS_n group. Hence $n \leq 6$.

The Proof of Theorem The conclusions (1) and (2) are obvious for $n = 1, 2$. For $n = 3$ we have $|G| = 2 \cdot 3$, so the conclusion (3) holds. For $n = 4$ we have $|G| = 2^2 \cdot 3$. Since G does not contain any subgroup of order 6, we infer that $G \cong A_4$.

Now we prove that OS_n groups ($n = 5, 6$) do not exist. Let G be an OS_5 group and $|G| = 2^2 \cdot 3 \cdot 5$. If G is not simple, then we may assume that N is a minimal normal subgroup of G and $|N| \geq 2$. Hence $|N| = 2, 3, 2^2$ or 5. Considering the semidirect product of N and the Sylow p-subgroup of G, where p is prime to $|N|$, we derive a contradiction. If G is simple, then $G \cong A_5$. Thus G contains a subgroup of order 12, a contradiction. Therefore OS_5 groups do not exist. Similarly we may prove that OS_6 groups do not exist also. The theorem is proved.

Comparing with Ref [1], here we do not use the classification of finite simple groups.

Problem *To classify all finite groups whose proper abelian subgroup orders are consecutive integers.*

References

[1] R.Brandl and Shi Wujie, *Finite groups whose element orders are consecutive integers*, J. Algebra, **143**(1991), 388—400.

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真子群阶为连续整数的有限群

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摘 要

本文回答了D.Macttale教授所提出的如下问题：确定所有的有限群，其真子群的阶为连续整数。

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Finite Groups whose Proper Subgroup Orders Are Consecutive Integers *

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In [1] the authors gave a complete classification of finite groups whose element orders are consecutive integers. After [1] was published, D.Macttale put forward the following problem: To classify all finite groups in which proper subgroup orders are consecutive integers. In this note we answer the above question and prove the following conclusion.

Theorem *Let G be a finite group and $\pi_s(G)$ a set of proper subgroup orders in G. If $\pi_s(G)=\{1,2,\cdots,n\}$, then $n\leq 4$ and one of the following holds.*

(1) $n=1$ and $G\cong Z_p$, p prime.

(2) $n=2$ and $G\cong Z_2\times Z_2$, or $G\cong Z_4$.

(3) $n=3$ and $G\cong S_3$, or $G\cong Z_6$.

(4) $n=4$ and $G\cong A_4$.

All groups discussed will be assumed to be finite and the notation is standard; see [4].

Lemma 1 *Let n be a positive integer. If $n\geq 7$, then there is the kth prime of the prime series satisfying the following inequalities.*

(1) $n\geq p_k>\frac{n}{2}$, and

(2) $p_1p_2\cdots p_{k-1}>n$, where $p_1=2, p_2=3,\cdots$, and p_i is the ith prime of the prime series.

Proof (1) is obtained immediately from Bertrand's postulate (see [3, Theorem 5.7.1]). And we may immediately check (2) for $n\leq 10$. While $n\geq 11$ we have $p_{k-2}\geq 5$ and the conclusion is similarly obtained from Bertrand's postulate.

For convenience, we call the groups G whose proper subgroup orders are consecutive integers OS_n group, where n is the maximal integer in $\pi_s(G)$.

Lemma 2 *Let G be a group. If G is an OS_n group, then $n\leq 6$.*

Proof Let p_k be the maximal prime and $p_k\leq n$. Since G is an OS_n group, $p_1p_2\cdots p_k \mid |G|$, where p_i is the ith prime of the prime series. Suppose P is a Sylow p_k-subgroup of G, then $|P|=p_k$ and P is cyclic by the assumption. Considering the normalizer $N_G(P)$ of P in G we have $|N_G(P)|\leq n$. If $n\geq 7$, then $N_G(P)=P$ from Lemma 1.(1) Therefore G has a normal p_k-complement H and $|H|\geq p_1p_2\cdots p_{k-1}$ from Burnside's theorem (see [2,

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Theorem 14.3.1]). Again from lemma 1.(2) we see that $|H| > n$. It is contrary to G being an OS_n group. Hence $n \leq 6$.

The Proof of Theorem The conclusions (1) and (2) are obvious for $n = 1, 2$. For $n = 3$ we have $|G| = 2 \cdot 3$, so the conclusion (3) holds. For $n = 4$ we have $|G| = 2^2 \cdot 3$. Since G does not contain any subgroup of order 6, we infer that $G \cong A_4$.

Now we prove that OS_n groups ($n = 5, 6$) do not exist. Let G be an OS_5 group and $|G| = 2^2 \cdot 3 \cdot 5$. If G is not simple, then we may assume that N is a minimal normal subgroup of G and $|N| \geq 2$. Hence $|N| = 2, 3, 2^2$ or 5. Considering the semidirect product of N and the Sylow p-subgroup of G, where p is prime to $|N|$, we derive a contradiction. If G is simple, then $G \cong A_5$. Thus G contains a subgroup of order 12, a contradiction. Therefore OS_5 groups do not exist. Similarly we may prove that OS_6 groups do not exist also. The theorem is proved.

Comparing with Ref [1], here we do not use the classification of finite simple groups.

Problem *To classify all finite groups whose proper abelian subgroup orders are consecutive integers.*

References

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