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$S(I)$ 的一个子半群

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摘 要

设I 为单位闭区间[0,1],$S(I)$ 为I 上所有连续自映射构成的半群. 本文研究了$S(I)$ 的一个子半群，讨论了这个子半群上的Green 关系以及某些理想和同余.

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A Subsemigroup of $S(I)$ *

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Abstract $S(I)$ is the semigroup of all continuous selfmaps of the unit closed interval $I = [0,1]$. This paper investigates a subsemigroup of $S(I)$ and discusses its Green's relations, some ideals and congruences.

Key words Semigroup, Green's relations, ideal, congruence.

1. Introduction

Let X be a topological space and $S(X)$ the semigroup of all continuous selfmaps of X. In the field of the theory of $S(X)$, many results have been achieved. However, till now, there remain many unsolved open problems. One of them is to determine the Green's relations for arbitrary elements of $S(X)$.

In this paper, the space under consideration will be the closed unit interval $I = [0,1]$. We endeavor to look for an appropriate subsemigroup of $S(I)$ to which a lot of irregular elements of $S(I)$ belong and on which the Green's relations can be perfectly determined. We attempt, in this way, to obtain some informations about the Green's relations for irregular elements of $S(I)$.

In Section 2, we decide a subsemigroup $S_1(I)$ of $S(I)$. And in Section 3, the Green's relations on $S_1(I)$ are characterized completely. Then, in Section 4 and 5, we investigate some ideals and congruences for $S_1(I)$, respectively.

2. The subsemigroup $S_1(I)$

First of all, we introduce some terminologies and symbols.

Definition 2.1 *A map $f \in S(I)$ is called elementary if there exists a division of I*

$$0 = a_0 < a_1 < \cdots < a_n = 1$$

such that every cut point a_i is a local extreme point of f and on every interval $[a_{i-1}, a_i]$ f is monotone. (Note in this paper the word "monotone" always means strictly monotone).

The interval $[a_{i-1}, a_i]$ is called the i-th monotone interval of f and the number of monotone intervals of f will be denoted by the symbol $M(f)$.

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The collection of all elementary surjections of $S(I)$ will be denoted by $S_1(I)$.

We denote the unit group of $S(I)$ by $G(I)$ which consists of all homeomorphisms from I onto itself. One easily verifies that $G(I) \subset S_1(I)$ and $M(f) = 1$ for each $f \in G(I)$.

We are now in a position to state the main result of this section.

Theorem 2.2 *$S_1(I)$ is a subsemigroup of $S(I)$.*

The key of proving this Theorem is to show that the product of two elementary maps is also elementary. To do this, we need primarily the following lemma:

Lemma 2.3 *Let $f, g \in S(I)$. Suppose g is monotone on $[a,b]$, f is monotone on $[c,d]$ and $g([a,b]) \subset [c,d]$. Then fg is monotone on $[a,b]$.*

Proof Suppose g is monotone increasing on $[a,b]$ and f is monotone increasing on $[c,d]$. Then for any $x, y \in [a,b]$ and $x < y$, we have $g(x) < g(y)$. Notice $g(x), g(y) \in [c,d]$ and f is increasing on $[c,d]$, then we know $fg(x) < fg(y)$, which means that fg is monotone increasing on $[a,b]$.

Similarly, we can show that in the other cases the conclusion is also true. □

The Proof of Theorem 2.2 Let $f, g \in S_1(I)$ and the divisions of f and g be

$$0 = a_0 < a_1 < \cdots < a_n = 1, \quad 0 = b_0 < b_1 < \cdots < b_m = 1,$$

respectively. Take any j $(1 \le j \le m)$ and denote $g([b_{j-1}, b_j])$ by $[c,d]$. If there are not any a_i in the open interval (c,d), then $[c,d] \subset [a_{i-1}, a_i]$ for some i $(1 \le i \le n)$, and it follows immediately from Lemma 2.3 that fg is monotone on $[b_{j-1}, b_j]$. Now suppose there are some cut points, $a_{i+1}, \cdots, a_{i+s}$, say, in the open interval (c,d). Let $g_j = g|[b_{j-1}, b_j]$, then one easily sees that g_j maps $[b_{j-1}, b_j]$ homeomorphically onto $[c,d]$. Let

$$b_{j1} = g_j^{-1}(a_{i+1}), \cdots, b_{js} = g_j^{-1}(a_{i+s}),$$

for convenience, we may suppose that g_j is increasing. Then $b_{j-1} < b_{j1} < \cdots < b_{js} < b_j$, and appealing to Lemma 2.3 again, fg is monotone on each of following intervals

$$[b_{j-1}, b_{j1}], [b_{j1}, b_{j2}], \cdots, [b_{js}, b_j].$$

Do the same things for every $[b_{j-1}, b_j]$ and we can obtain a division $0 = c_0 < c_1 < \cdots < c_t = 1$, such that fg is monotone on each $[c_{i-1}, c_i]$.

Now let us observe each c_i $(1 \le i \le t)$. If fg is monotone on $[c_{i-1}, c_{i+1}]$ then reject c_i from the division, otherwise, reserve it. In this way, we can obtain a new division

$$0 = d_0 < d_1 < \cdots < d_s = 1.$$

The new division coming from the former one satisfies that each d_i is a local extreme point of fg and on each interval $[d_{i-1}, d_i]$ fg is monotone. It follows that fg is elementary.

In addition, both f and g are surjective and so is fg, that is, $fg \in S_1(I)$. The proof is now completed. □

Denote by R the subset of $S(I)$ consisting of all surjections which are not constant on subintervals of I. It is well known that R forms a subsemigroup of $S(I)$ [4]. Obviously,

$S_1(I) \subset R$. However, the contrary is not true. Here we point out $R \not\subset S_1(I)$. For example, let $f: I \to I$ be defined as follows

$$f(x) = \begin{cases} \frac{1}{2} & x = 0 \\ x\sin\frac{1}{x} + \frac{1}{2} & 0 < x \le \frac{1}{\pi} \\ \frac{\pi}{\pi - 2}(x + \frac{1}{2} - \frac{2}{\pi}) & \frac{1}{\pi} < x \le \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \le x \le 1. \end{cases}$$

It is easy to check that f is continuous and not constant on any subinterval of I while f maps I onto I, that is, $f \in R$. But $f \notin S_1(I)$ since f has infinitely many local extreme points and infinitely many monotone intervals.

Theorem 2.4 *Let $f \in S_1(I)$, then f is regular in $S_1(I)$ if and only if $f \in G(I)$.*

Proof The sufficiency is obvious, we only need show the necessity. Let f be regular in $S_1(I)$. Then there exists some $g \in S_1(I)$ such that $fgf = f$. For any $x \in I$, we may take some $y \in I$ such that $f(y) = x$ since f is surjective. Therefore $fg(x) = fgf(y) = f(y) = x$, that is, $fg = id$ (where id is the identity map on I). This implies that f is a homeomorphism, i.e., $f \in G(I)$. □

It is well known that $f \in S(I)$ is regular if and only if f maps some subinterval of I homeomorphically onto his image $f(I)$ [1]. In view of Theorem 2.4, we know that $S_1(I)$ is not a regular semigroup. Yet we have to notice here that the irregular elements of $S_1(I)$ may be regular in $S(I)$. However, undoubtedly, in $S_1(I)$ there are large quantities of irregular elements of $S(I)$.

3. The Green's relations on $S_1(I)$

Theorem 3.1 *Let $f, g \in S_1(I)$, then $f\mathcal{L}g$ if and only if there exists a unique $h \in G(I)$ satifying $hf = g$.*

Proof We need only to show the necessity. Suppose $f\mathcal{L}g$, then there exist $h, k \in S_1(I)$ such that $hf = g$ and $kg = f$. Thus, $khf = f$. For any $x \in I$, let $x = f(y)$ for some $y \in I$. Then $kh(x) = khf(y) = f(y) = x$. This means $kh = id$, moreover, $h, k \in G(I)$ and $k = h^{-1}$.

If $h, h_1 \in G(I)$ satisfy $hf = g$ and $h_1 f = g$, then $h_1(x) = h_1 f(y) = g(y) = hf(y) = h(x)$, that is, $h_1 = h$. □

Corollary 3.2 *Let $f, g \in S_1(I)$ and $f\mathcal{L}g$. Then $M(f) = M(g)$, moreover, f and g have the same division.*

Proof Let $h \in G(I)$ such that $hf = g$ and let the division of g be

$$0 = b_0 < b_1 < \cdots < b_n = 1.$$

Without loss of generality, suppose h is increasing. Denote $J_i = [b_{i-1}, b_i]$ $(1 \le i \le n)$.

Suppose $g = hf$ is increasing on J_1, then obviously so is f by Lemma 2.3. Furthermore, f is decreasing on J_2 just as g is and so on. Consequently, all J_1 are monotone intervals of f while all b_i are local extreme points of f and the conclusion follows immediately. □

Before considering Green's $\mathcal{R}$ relation on $S_1(I)$, we establish a lemma.

Lemma 3.3 *Suppose $f, g, h \in S_1(I)$ and $fh = g$, then $M(g) \geq M(f)$.*

Proof Let $J_1, J_2, \ldots, J_n$ be the monotone intervals of g. Then fh is injective on each J_i. Furthermore, h is injective on J_i and f is injective on $h(J_i)$ for each i $(1 \leq i \leq n)$. Hence each $h(J_i)$ belongs to some monotone interval of f. Notice that h is surjective,

$$\bigcup_{i=1}^{n} h(J_i) = h(\bigcup_{i=1}^{n} J_i) = h(I) = I.$$

Therefore, f has at most n monotone intervals, that is, $M(f) \leq M(g)$. □

Theorem 3.4 *Let $f, g \in S_1(I)$, then $f\mathcal{R}g$ if and only if there exists $h \in G(I)$ such that $fh = g$.*

Proof We only need to show the necessity. Suppose $f\mathcal{R}g$, then there exist $h, k \in S_1(I)$ such that $fh = g$ and $gk = f$. It follows immediately from Lemma 3.3 that $M(g) \geq M(f)$ and $M(f) \geq M(g)$. Thus, $M(f) = M(g)$.

Let $J_1, J_2, \ldots, J_n$ be all the monotone intervals of g. Then h is injective on each J_i because of $g = fh$. Next, we are going to show that h has the same monotonicity on each J_i which will imply $h \in G(I)$. We may assume that h is monotone increasing on J_1. If h is monotone decreasing on J_2, then $h(J_1)$ and $h(J_2)$ are both nondegenerate closed intervals and they have the common right end point. Consequently, $h(J_1) \subset h(J_2)$ or $h(J_1) \supset h(J_2)$ and therefore, $h(J_1) \cup h(J_2) = h(J_1)$ or $h(J_2)$. Notice f is monotone on each $h(J_i)$ and that $\bigcup_{i=1}^{n} h(J_i) = I$, so f has at most $n-1$ monotone intervals. Thus $M(f) \leq n-1 < n = M(g)$ which is obviously a contradiction. Therefore, h is monotone increasing on J_2 too.

Similarly, h is monotone increasing on every J_i, this means h is monotone increasing on all I and h is injective. Note I is compact and Hausdorff, and from this we know that h is a homeomorphism. □

Remark J. Mioduszewski [4] arrived at a similar result for the subsemigroup R of $S(I)$ mentioned above, and his result includes Theorem 3.4 of this paper, but his proof is too complicated for us to accept. For the sake of completeness, it is necessary for us to put forward the result for $S_1(I)$ and give a concise proof.

Definition 3.5 *Let $f, g \in S_1(I)$ and their monotone intervals be $J_1, \cdots, J_m$ and $K_1, \cdots, K_m$ respectively. If for each i $(1 \leq i \leq m)$, $f(J_i) = g(K_i)$ and the monotonicities of f on J_i and of g on K_i are identical, then we say that f and g are similar. If for each i $(1 \leq i \leq m)$, $f(J_i) = g(K_{m-i+1})$ and the monotonicities of f on J_i and of g on K_{m-i+1} are contrary, then we say that f and g are dual-similar.*

The next result gives another characterization of the Green's $\mathcal{R}$ relation on $S_1(I)$.

Theorem 3.6 *Let $f, g \in S_1(I)$, then $f\mathcal{R}g$ if and only if f and g are similar or dual-similar.*

Proof Suppose $f\mathcal{R}g$ then there exists $h \in G(I)$ such that $fh = g$, and according to the proof of Theorem 3.4, $M(f) = M(g)$. Let $0 = a_0 < a_1 < \cdots < a_n = 1, 0 = b_0 < b_1 < \cdots < b_n = 1$ be the divisions of f and g, respectively. Denote $J_i = [a_{i-1}, a_i], K_i = [b_{i-1}, b_i], 1 \le i \le n$. Then $h(K_1), h(K_2), \cdots, h(K_n)$ must be all the monotone intervals of f.

If h is increasing, then $h(K_i) = J_i, f(J_i) = fh(K_i) = g(K_i)$ and the monotonicities of f on J_i and of g on K_i are identical for each i, that is, f and g are similar.

If h is decreasing, then $h(K_{n-i+1}) = J_i, f(J_i) = fh(K_{n-i+1}) = g(K_{n-i+1})$ while the monotonicities of f on J_i and of g on K_{n-i+1} are contrary for each i. That means f and g are dual-similar.

On the other hand, if f and g are similar, let $f_i = f|J_i, g_i = g|K_i$ (here J_i and K_i mean the same as above). Now define $h : I \to I$ by

$$h(x) = \begin{cases} f_1^{-1}g_1(x) & x \in K_1 \\ f_2^{-1}g_2(x) & x \in K_2 \\ \cdots & \cdots \\ f_n^{-1}g_n(x) & x \in K_n. \end{cases}$$

It is easy to see that h is continuous and surjective. Moreover, notice that g_i, f_i and f_i^{-1} have the same monotonicity and by Lemma 2.3, $f_i^{-1}g_i$ is monotone increasing on K_i for each i. That means h is an increasing homeomorphism. Obviously, $fh = g$, so $f\mathcal{R}g$.

If f and g are dual-similar, then h can be defined as

$$h(x) = \begin{cases} f_n^{-1}g_1(x) & x \in K_1 \\ f_{n-1}^{-1}g_2(x) & x \in K_2 \\ \cdots & \cdots \\ f_1^{-1}g_n(x) & x \in K_n \end{cases}$$

and in the similar manner we can show that h is a decreasing homeomorphism and $fh = g$, here again we have $f\mathcal{R}g$. □

We have seen that if $f\mathcal{L}g$ then the homeomorphism h satisfying $hf = g$ is unique. Naturally, we want to know what it is like in case of $f\mathcal{R}g$. In order to clear up this point, we need a torminology at first.

Definition 3.7 *$f \in S_1(I)$ is called symmetric if $M(f)$ is even, say $2n$, and $f(J_i) = f(J_{2n-i+1})$ for each monotone interval J_i of f.*

Otherwise, f is called non-symmetric.

According to Theorem 3.6, it is easy to verify that f is symmetric if and only if g is symmetric whenever $f\mathcal{R}g$.

Theorem 3.8 *Suppose $f, g \in S_1(I)$ and $f\mathcal{R}g$. If f is non- symmetric, then there exists a unique $h \in G(I)$ satisfying $fh = g$. If f is symmetric, then there exist exactly two $h \in G(I)$ satisfying $fh = g$, one of them is increasing and the other is decreasing.*

Proof Let $h, k \in G(I)$ such that $fh = g = fk$, then $fhk^{-1} = f$ and $hk^{-1} \in G(I)$.

When f is non-symmertric, then there are two possible cases.

Case 1 $M(f)$ is odd, say $2n+1$.

We assert that hk^{-1} must be increasing. Otherwise, suppose hk^{-1} is decreasing, then it will map $J_1 = [a_0, a_1]$, the first monotone interval of f, homeomorphically onto $J_{2n+1} = [a_{2n}, a_{2n+1}]$, while $hk^{-1}(a_0) = hk^{-1}(0) = 1 = a_{2n+1}$, $hk^{-1}(a_1) = a_{2n}$.

Since the monotonicity of an elementary map changes alternatively, so $f_1 = f|J_1$ and $f_{2n+1} = f|J_{2n+1}$ have the same monotonicity. However,

$$f(a_0) = fhk^{-1}(a_0) = f(a_{2n+1}), \quad f(a_1) = fhk^{-1}(a_1) = f(a_{2n}).$$

Here arises a contradiction: the monotonicity of f_1 is contrary to that of f_{2n+1}. Thus, our assertion holds. Consequently, $hk^{-1}(J_i) = J_i$ for each i. In addition, $fhk^{-1}(x) = f(x)$ for each $x \in I$ and that f is injective on each J_i, hence $hk^{-1}(x) = x$ for each $x \in I$ that means $hk^{-1} = id$ and $k = h$.

Case 2 $M(f) = 2n$, but there is some t $(1 \le t \le 2n)$ such that $f(J_t) \ne f(J_{2n-t+1})$.

In the similar way we can show that hk^{-1} must also be increasing and $hk^{-1} = id, k = h$. Now we have seen that $h \in G(I)$ satisfying $fh = g$ is unique when f is non-symmetric.

If f is symmetric, it is easy to see that hk^{-1} will be either increasing or decreasing. When hk^{-1} is increasing we can see $hk^{-1} = id$ and $k = h$ as above. Now suppose hk^{-1} is decreasing, then $hk^{-1}(J_i) = J_{2n-i+1}$ for each i. Note $f(J_i) = f(J_{2n-i+1})$ and f is injective on each J_i, we can define $u : I \to I$ as

$$u(x) = \begin{cases} f_{2n}^{-1} f_1(x) & x \in J_1 \\ f_{2n-1}^{-1} f_2(x) & x \in J_2 \\ \cdots & \cdots \\ f_1^{-1} f_{2n}(x) & x \in J_{2n} \end{cases}$$

Then it is easy to verify that u is a decreasing homeomorphism uniquely determined by f, and that $u^2 = id$, namely u is an involution. Furthermore, we can also verify $hk^{-1} = u$. Thus, $h = uk$ and $k = uh$. Then, we have seen that when f is symmetric there are exactly two homeomorphisms satisfying $fh = g$. If h is increasing then $k = uh$ is decreasing, otherwise, $k = uh$ is increasing. The proof is complete. □

Consequently, we can determine the Green's $\mathcal{H}$ and $\mathcal{D}$ relations. The proofs of the next two results are routine. We omit the details. □

Theorem 3.9 *Let $f, g \in S_1(I)$. Then $f\mathcal{H}g$ if and only if there exist $h, k \in G(I)$ such that $fh = kf = g$.* □

Theorem 3.10 *Let $f, g \in S_1(I)$. Then $f\mathcal{D}g$ if and only if there exist $h, k \in G(I)$ such that $f = hgk$.* □

Before discussing Green's $\mathcal{J}$ relation on $S_1(I)$, let us see the following lemma.

Lemma 3.11 *Let $f, g, h \in S_1(I)$, $f = hg$ and $M(f) = M(g)$. Then $h \in G(I)$.*

Proof Let $J_1, J_2, \cdots, J_n$ be all the monotone intervals of f. Then g is injective (i.e., monotone) on each J_1. Note $M(f) = M(g)$, so $J_1, J_2, \cdots, J_n$ are precisely all the monotone intervals of g. Denote $T_i = g(J_i)$, $1 \le i \le n$. Obviously, each T_i is nondegenerate closed interval on which h is injcetive (i.e. monotone). Since the monotonicities of g change alternatively, so $T_i \subset T_{i+1}$ or $T_i \subset T_{i+1}$ for each i. Then h is monotone on $T_i \cup T_{i+1}$. Furthermore, h is monotone on

$$\bigcup_{i=1}^{n} T_i = \bigcup_{i=1}^{n} g(J_i) = g(\bigcup_{i=1}^{n} J_i) = g(I) = I.$$

It follows that $h \in G(I)$.

Theorem 3.12 *Let $f, g \in S_1(I)$. Then $f J g$ if and only if there exist $h, k \in G(I)$ such that $f = hgk$. Consequently, $\mathcal{J}$ and $\mathcal{D}$ coincide on $S_1(I)$.*

Proof It is enough to show only the necessity. First, by using the same method in Lemma 3.3 we can show the following assertion: If $f = hgk$ for some $f, g, h, k \in S_1(I)$, then $M(f) \ge M(g)$.

Let $f J g$, then there exist $h, k, u, v \in S_1(I)$ such that $f = hgk$ and $g = ufv$. By the assertion just mentioned above, we have $M(f) \ge M(g)$ and $M(f) \le M(g)$, that is, $M(f) = M(g)$. Let $J_1, J_2, \cdots, J_n$ be all the monotone intervals of f, then $M(f) = M(g)$ makes it sure that $k(J_1), k(J_2), \cdots, k(J_n)$ are all the monotone intervals of g. Furthermore, through the use of the way of Theorem 3.4, we can see $k \in G(I)$.

Denote $g_1 = gk$, then $M(g_1) = M(g) = M(f)$ and $f = hgk = hg_1$. In view of Lemma 3.11 we know $h \in G(I)$. □

Theorem 3.13 *For each $f \in S_1(I)$, the Green's $\mathcal{L}$- class, $\mathcal{R}$- class, $\mathcal{H}$-class, $\mathcal{D}$-class and $\mathcal{J}$-class containing f are $L_f = G(I)f$, $R_f = fG(I)$, $H_f = G(I)f \cap fG(I)$, $D_f = J_f = G(I)fG(I)$, respectively.* □

Consequently, the Green's $\mathcal{L}$-class, $\mathcal{R}$- class, $\mathcal{H}$-class, $\mathcal{D}$-class and $\mathcal{J}$-class containing the identity map all coincide with $G(I)$. Therefore, $G(I)$ is the only regular D-class of $S_1(I)$.

4. The ideals of $S_1(I)$

The symbols D_n are denoted as $D_n = \{f \in S_1(I) : \ M(f) \ge n\}$, $n = 1, 2, \cdots$. It is easy to see that $D_1 = S_1(I)$, $D_2 = S_1(I) - G(I)$ and $D_1 \supset D_2 \supset D_3 \supset \cdots$

Theorem 4.1 *The maximal left ideal, the maximal right ideal and the maximal two-sided ideal all coincide in the semigroup $S_1(I)$. Precisely, the maximal (left, right and two-sided) ideal is D_2.*

Proof It is well known that if $fg \in G(I)$, then both f and g belong to $G(I)$ for any $f, g \in S(I)$. From this we can easily see that D_2 is a left ideal as well as right ideal and hence a two-sided ideal of $S_1(I)$.

Now suppose E is a left ideal of $S_1(I)$ such that $D_2 \subset E$, then there exists some $f \in E \cap G(I)$ and $id = f^{-1}f \in S_1(I)E \subset E$.

Therefore, $S_1(I) \subset E$ and $E = S_1(I)$. This means D_2 is the maximal left ideal of $S_1(I)$. Similarly, we can see that D_2 is the maximal right ideal and the maximal two-sided ideal of $S_1(I)$ as well. □

Lemma 4.2 *Let $f, g \in S_1(I)$. Then every local extreme point of g must be the local extreme point of fg.*

Proof Let the division of g be $0 = b_0 < b_1 < \cdots < b_n = 1$. We need only show that each of $b_1, b_2, \cdots, b_{n-1}$ is a local extreme point of fg.

Now take any b_i $(1 \le i \le n-1)$, without loss of generality, let us assume that b_i is a local maximum point of g. Then $g(b_i) \ne 0$ and there exists $\varepsilon > 0$ such that $g(x) < g(b_i)$ for any $x \in (b_i - \varepsilon, b_i + \varepsilon) - \{b_i\}$. There are three cases that will be discussed here.

Case 1 $g(b_i) = a_k$ is a local maximum point of f. Since g is continuous, we can choose $\delta > 0$ such that $\delta < \varepsilon$ and $g((b_i - \delta, b_i + \delta)) \subset [a_{k-1}, a_k]$. Here $[a_{k-1}, a_k]$ is a monotone increasing interval of f. Then, for arbitrary $x \in (b_i - \delta, b_i + \delta) - \{b_i\}$, $g(x) < g(b_i)$ and $fg(x) < fg(b_i)$, b_i is a local maximum point of fg.

Case 2 $g(b_i) = a_k$ is a local maximum point of f. Similarly, we can take $\delta > 0$ such that $\delta < \varepsilon$ and $g((b_i - \delta, b_i + \delta)) \subset [a_{k-1}, a_k]$. Here $[a_{k-1}, a_k]$ is a monotone decreasing interval of f. Then, for arbitrary $x \in (b_i - \delta, b_i + \delta) - \{b_i\}$, $g(x) < g(b_i)$ and $fg(x) > fg(b_i)$, b_i is a local minimum point of fg.

Case 3 $g(b_i)$ is not an extreme point of f. Then there exists a monotone interval of $f, [a_{k-1}, a_k]$ say, such that $g(b_i) \in (a_{k-1}, a_k)$. Again, by the continuity of g we can take $\delta > 0$ such that $\delta < \varepsilon$ and $g((b_i - \delta, b_i + \delta)) \subset (a_{k-1}, a_k)$. Then in the similar way of case 1 or case 2, we can also see that b_i is a local maximum or local minimum point of fg according as f is increasing or decreasing on the interval $[a_{k-1}, a_k]$. The proof is completed. □

Lemma 4.3 $M(fg) \ge \max\{M(f), M(g)\}$ *for any* $f, g \in S_1(I)$.

Proof The previous lemma tells us that $M(fg) \ge M(g)$. What remains to be done is to show $M(fg) \ge M(f)$.

Let the division of g be $0 = b_0 < b_1 < \cdots < b_n = 1$ and denote $B = \{g(b_i) : 0 \le i \le n\}$. Obviously, B has at most $n+1$ points. Let the division of f be $0 = a_0 < a_1 < \cdots < a_m = 1$ and suppose $a_{j1}, a_{j2}, \cdots, a_{jk}$ are all the cut points of the division of f belonging to B. Now let us rename the remaining $m - k + 1$ cut points of the division of f as $c_1 < c_2 < \cdots < c_{m-k+1}$.

We assert that each c_s $(1 \le s \le m - k + 1)$ determines at least one local extreme point d_s of fg and $d_s \ne d_t$ when $s \ne t$.

In fact, let $T_i = g([b_{i-1}, b_i])$, $1 \le i \le n$ as above, then each T_i is a nondegenerate closed interval and $\bigcup_{i=1}^{n} T_i = I$. For each c_s, there exists at least one T_i such that c_s is an interior point of T_i. Let $g_i = g|[b_{i-1}, b_i]$ and $d_s = g_i^{-1}(c_s)$, then d_s is an interior point of $[b_{i-1}, b_i]$. Notice c_s is a local extreme point of f and by Lemma 2.3, d_s must be a local extreme point of fg.

For distinct s and t, if $d_s \in (b_{i-1}, b_i)$, $d_t \in (b_{j-1}, b_j)$ and $i \ne j$, then obviously $d_s \ne d_t$. If d_s and d_t belong to the same (b_{i-1}, b_j), then c_s and c_t belong to the interior of T_i. Since g_i is injective and $c_s \ne c_t$, then we also have $d_s \ne d_t$. Thus our assertion holds.

Then the number of local extreme points of fg is at least $(n+1)+(m-k+1) \geq m+1$. This means $M(fg) \geq M(f)$. □

By virtue of Lemma 4.3 we get the main result of this section immediately.

Theorem 4.4 *For every natural number n, D_n is an ideal (i.e., two-sided ideal) of $S_1(I)$ and $D_1, D_2, \cdots$ form a decreasing chain of ideals.* □

5. Some congruences on $S_1(I)$

Let $\sigma_0 = \{(f,f) : f \in S_1(I)\}$ and $\sigma_n = (D_n \times D_n) \cup \sigma_0$, $n = 1,2,\cdots$, then from Theorem 4.4 we get

Theorem 5.1 *For every natural number n, σ_n is one of Rees congruences on $S_1(I)$ and $\sigma_1, \sigma_2, \cdots$ form a decreasing chain of Rees congruences on $S_1(I)$.* □

Besides Rees congruences we are going to consider some other congruences on $S_1(I)$.

From [3, p255] the normal subgroups of $G(I)$ are precisely the groups $G(I)$, F, Q and $\{id\}$, where F is the group of all increasing homeomorphisms and Q denotes all those homeomorphisms in $G(I)$ which coincide with the identity map in a neighborhood of 0 and in a neighborhood of 1. Evidently, $Q \subset F$. Denote

$$\tau_0 = \{(f,f) : f \in G(I)\}, \qquad tau_1 = \{(f,g) \in G(I) \times G(I) : fg^{-1} \in Q\},$$

$$\tau_2 = \{(f,g) \in G(I) \times G(I) : fg^{-1} \in F\}, \quad \tau_3 = G(I) \times G(I).$$

Then $\tau_0, \tau_1, \tau_2, \tau_3$ are the only four congruences on $G(I)$. Denote $\rho_1 = \sigma_2 \cup \tau_1$, $\rho_2 = \sigma_2 \cup \tau_2$, $\rho_3 = \sigma_2 \cup \tau_3$. Obviously, $\sigma_2 \subset \rho_1 \subset \rho_2 \subset \rho_3$. Furthermore, we have

Theorem 5.2 *ρ_1, ρ_2 and ρ_3 are the only proper congruences on $S_1(I)$ properly containing σ_2. Consequently, ρ_3 is the greatest proper congruence on $S_1(I)$ containing σ_2.*

Proof It is easy to verify that all ρ_1, ρ_2 and ρ_3 are congruences on $S_1(I)$. Now suppose σ is a congruence on $S_1(I)$ satisfying $\sigma \supset \sigma_2$, $\sigma \neq \sigma_2$ and $\sigma \neq \rho_i$, $i = 1,2,3$, we have to show that σ must be the universal congruence $S_1(I) \times S_1(I)$.

It readily follows that there must exist some $f \in G(I)$ and $g \notin G(I)$ such that $(f,g) \in \sigma$. For any $h \in G(I)$, let $k = hf^{-1}$, then $k \in G(I)$. Thus, $(kf, kg) = (h, kg) \in \sigma$. Since $kg \in D_2$, then $(kg, g) \in \sigma_2 \subset \sigma$. Therefore, $(h,g) \in \sigma$ holds for any $h \in G(I)$. Now for arbitrary $u, v \in S_1(I)$, there are three cases to consider.

Case 1 $u, v \in G(I)$. Then $(u,g) \in \sigma$ and $(v,g) \in \sigma$ which implies $(u,v) \in \sigma$.

Case 2 $u \notin G(I)$ and $v \notin G(I)$. Obviously in this case $(u,v) \in \sigma_2 \subset \sigma$.

Case 3 $u \in G(I)$ and $v \notin G(I)$. Then $(u,g) \in \sigma$, $(g,v) \in \sigma_2 \subset \sigma$. We also have $(u,v) \in \sigma$.

Therefore, $\sigma = S_1(I) \times S_1(I)$. The remaining assertion is obvious. □

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$S(I)$ 的一个子半群

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摘 要

设I 为单位闭区间[0,1],$S(I)$ 为I 上所有连续自映射构成的半群. 本文研究了$S(I)$ 的一个子半群，讨论了这个子半群上的Green 关系以及某些理想和同余.