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The Integral Formula of Pontrjagin Characteristic Form on a Grassmann Manifold *

Mei Xiangming
(Dept. of Math., Capital Normal University, Beijing 100037)

Abstract Let $G_{n+m,n} = \frac{O(n+m)}{O(n)\times O(m)}$ be a Grassmann manifold. $Z_{2k} = [\underbrace{m-2,\cdots,m-2}_{2k}, m,\cdots,m]$ is a Schubert variety of $G_{n+m,n}$. Let E be the canonical vector bundle on $G_{n+m,n}$, and $E\otimes C$ is the complexification of E. Let $\tilde{E}$ be the associated unitary $(n-2k)$-frame bundle of $E\otimes C$, and E' is a subbundle of $E\otimes C$. In this paper, we prove the following integral formula:

$$c_{4k}\cdot Z_{2k} = (-1)^k\int_{c_{4k}} P_k(\Omega) - \int_{\partial c_{4k}} u^*\tilde{\tau} - \int_{\partial c_{4k}} u^*\tau',$$

where c_{4k} is a $4k$-dimemsional chain of $G_{n+m,n}$, and $P_k(\Omega)$ is the kth Pontrjagin characteristic form of $G_{n+m,n}$, $\tilde{\tau}$ and τ' are the $(4k-1)$-forms defined on $\tilde{E}$ and E' respectively.

Keywords Grassmann manifold, Schubert variety, Pontrjagin characteristic form

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1. Introduction

In 1944, S.S.Chern [1] gave an intrinsic proof of the Gauss-Bonnet formula on a compact Riemannian maniford, in fact, he gave an integral formula of the Euler characteristic form. In 1959, J.Eells [2] generalized the result of Chern, he gave the integral formula of the Stiefel-Whitney characteristic classes. In 1976, K.L.Wu [3] also gave the integral formula of the Chern characteristic forms. In this paper, we will give further the integral formulas of the Pontrjagin characteristic forms. But our method is different to [3], we needn't use the local charts of the Grassmann manifold, so that it ensures the global existence of the vector fields appearing in this paper, and overcomes the defects of [3].

We discuss only in real Grassmann manifold, and denoted it by $G_{n+m,n}$.

2. The cell decomposition of Grassmann manifold

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Let R^{n+m} be $(n+m)$-dimensional Euclidean space, the Euclidean metric is

$$(x,y)=\sum_{A=1}^{n+m} x^A y^A,\quad x^A, y^A \in R,$$

and $O(n+m)$ is the orthogonal group of R^{n+m}.

The Grassmann manifold $G_{n+m,n}$ is the totality of all n-planes of R^{n+m} passing through the origin, it is a nm-dimensinal compact symmetric homogeneus space. On $G_{n+m,n}$, there exists a cannonical vector boundle E, its fibre at $x \in G_{n+m,n}$ is just the n-plane x of R^{n+m}. Let $E\otimes C$ be the complexification of E, it is a complex vector bundle, its fibre at $x \in G_{n+m,n}$ is the complex n-plane $x \otimes C$. Introduces a Hermitian metric in the fibres of the complex vector bundle $E \otimes C$ such that

$$(x_C, y_C)=\sum_{A=1}^{n+m} x_C^A \cdot y_C^{-A},\quad x^A, y^A \in C,$$

it endows $E \otimes C$ to be Hermitian vector bundle. Finnally, let $\tilde{E}$ be the unitary $(n-2k)$-frame bundle associated to $E \otimes C$, its fibre at $x \in G_{n+m,n}$ is the totality of all unitary $(n-2k)$-frames on the complex n-plane $x \otimes C$.

Give an ordered sequencec ω of non-negative integers:

$$0 \le \omega(1) \le \cdots \le \omega(i) \le \cdots \le \omega(n) \le m, \tag{1}$$

it determines a sequence of subspaces of R^{n+m}:

$$0 \subset E_1 \subset \cdots \subset E_i \subset \cdots \subset E_n \subset R^{n+m}, \tag{2}$$

where dim $E_i = i + \omega(i), (i = 1, \cdots, n)$. Choose an orthonormal basis of R^{n+m}

$$e_1, \cdots, e_n, f_1, \cdots, f_m,$$

such that $e_1, \cdots, e_i, f_1, \cdots, f_{\omega(i)}$ are the orthonormal basis of the subspace E_i.

The sequence of the subspaces (2) determines a Schubert variety of $G_{n+m,n}$:

$$Z_\omega = \{x \in G_{n+m,n} : \dim(x \cap E_i) \ge i\},$$

since the sequence of subspaces (2) is determined by the sequence of non- negative integers (1), so that Z_ω can be also denoted by $[\omega(1)\cdots,\omega(i),\cdots;,\omega(n)]$. Z_ω is a pseudomanifold of dimension $\sum_{i=1}^n \omega(i)$, the totality of Z_ω's constitute the cell decomposition of $G_{n+m,n}$ [5].

Consider the following Schubert variety:

$$Z_{2k} = [\underbrace{m-2,\cdots,m}_{2k}-2, m,\cdots,m] = \{x \in G_{n+m,n} : \dim(x \cap E_i) \ge 1\},$$

where $\dim E_i = i + (m-2)$, when $1 \le i \le 2k$; $\dim E_i = i + m$, when $2k+1 \le i \le n$. Z_{2k} is a $(nm-2k)$-dimensional cycle of $G_{n+m,n}$. The crucial condition to determine Z_{2k} is

$$\dim(x \cap E_{2k}) \ge 2k,$$

where the orthonormal basis of E_{2k} are $\{e_1, \cdots, e_{2k}, f_1, \cdots, f_{m-2}\}$. If $x \notin Z_{2k}$, then

$$\dim(x \cap E_{2k}) \geq 2k - 1,$$

then the dimension of the projection of $E_{2k}^{\perp}$ onto x along E_{2k} is greater that $n - 2k + 1$. Let the images of the basis of $E_{2k}^{\perp}$ under above projection be $\{e'_{2k+1}, \cdots, e'_n, f'_{m-1}, f'_m\}$, at least $(n - 2k + 1)$ of them are linearly independent, so that there exists $(n - 2k + 1)$ linearly independent complex vectors on the complex n-plane $x \otimes C$:

$$f'_{m-1} + \sqrt{-1} f'_m, f'_m + \sqrt{-1} e'_n, e'_n + \sqrt{-1} e'_{n-1}, \cdots, e'_{2k+2} + \sqrt{-1} e'_{2k+1},$$

using the Hermitian metric to orthonormalize theorem, we obtain an unitary $(n - 2k + 1)$ frame fields $\{u_1, \cdots, u_{n-2k}, \tilde{u}\}$ on $G_{n+m,n} \backslash Z_{2k}$. Similarly, we can obtain an unitary $(n - 2k)$ frame field $u = \{u_1, \cdots, u_{n-2k}\}$ on $G_{n+m,n} \backslash Z_{2k+1}$, and an unit vector field $\tilde{u}$ orthogonal to u with zero points in Z_{2k}. Then we get

Theorem 1 *There exists an unitary $(n - 2k + 1)$ frame fiele $\{u, \tilde{u}\}$ on $G_{n+m,n} \backslash Z_{2k}$, and an unitary $(n - 2k)$ frame field u on $G_{n+m,n} \backslash Z_{2k+1}$, and an unit vector field $\tilde{u}$ orthogonal to u on $G_{n+m,n} \backslash Z_{2k+1}$ with the zero set $(G_{n+m,n} \backslash Z_{2k+1}) \cap Z_{2k}$.*

3. Pontrjagin characteristic form and its transgressive form

Let $\{e_1, \cdots, e_{n+m}\}$ be the moving frames of Eucliddean $(n + m)$-space R^{n+m}, they are the elements of the orthogonal group $O(n + m)$. The infinitesmal displacement of the moving frames are

$$de_A = \sum_{B=1}^{n+m} \omega_{AB} e_B, \quad (A = 1, \cdots, n + m)$$

$(\omega_{AB})(A, B = 1, \cdots, n + m)(\omega_{AB} + \omega_{BA} = 0)$ are Maurer-Cartan forms of $O(n + m)$, they satisfy the equation of structure:

$$d\omega_{AB} = \sum_{C=1}^{n+m} \omega_{AC} \wedge \omega_{CB}, \quad (A, B = 1, \cdots, n + m).$$

Choose the first n vectors $\{e_1, \cdots, e_n\}$ of the moving frames to be the local sections of the vector bundle E, then the canonical connection ∇ of E is defined by

$$\nabla e_i = \sum_{j=1}^{n} \omega_{1j} e_j, \quad (i = 1, \cdots, n)$$

the matrix of the connection forms of ∇ is $\omega = (\omega_{ij})(i, j = 1, \cdots, n)$, the corresponding matrix of the curvature form is $(\Omega_{ij})(i, j = 1, \cdots, n)$, where

$$\Omega_{ij} = -\Omega_{ji} = d\omega_{ij} - \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj} = \sum_{s=n+1}^{n+m} \omega_{is} \wedge \omega_{sj}.$$

The kth Pontrjagin characteristic form $P_k(\Omega)$ of the vector bundle E is defined by ($k = 1,\cdots,[\frac{n}{2}]$)

$$P_k(\Omega) = \frac{1}{(2\pi)^{2k}\cdot 2k!}\sum_{i,j=1}^{n}\epsilon_{i_1,\cdots,i_{2k};j_1,\cdots,j_{2k}}\Omega_{i_1j_1}\wedge\cdots\wedge\Omega_{i_{2k}j_{2k}},$$

t is a closed $4k$-form of the base space $G_{n+m,n}$ of the vector bundle E, so that we can also call it the kth Pontrjagin characteristic form of the Grassmann manifold $G_{n+m,n}$.

The canonical connection ∇ of the vector bundle E induces a connection ∇^C on the Hermitian vector bundle $E\otimes C$, the matrix of the connection forms of ∇^C is the same as ∇, i.e., $\omega^C = \omega = (\omega_{ij})$, so that the corresponding matrix of the curvature forms is also she same as Ω, i.e. $\Omega^C = (\Omega_{ij})$. The kth Chern characteristic form of the complex vector oundle $E\otimes C$ is defined by ($k = 1,\cdots,n$)

$$C_k(\Omega^C) = \frac{1}{(2\pi i)^k\cdot k!}\sum_{i,j=1}^{k}\epsilon_{i_1,\cdots,i_k;j_1,\cdots,j_k}\Omega_{i_1j_1}\wedge\cdots\wedge\Omega_{i_kj_k}.$$

$C_k(\Omega^C)$ is a real closed $2k$-form of the Grassmann manifold $G_{n+m,n}$, it is obviously that

$$P_k(\Omega) = (-1)^k C_{2k}(\Omega^C).$$

Consider the subbundle E' of the complex vector bundle $E\otimes C$ on $G_{n+m,n}\backslash Z_{2k+1}$. According to the Lemma 1, there exists an unitary $(n-2k)$ fram field $u = \{u_1,\cdots,u_{n-2k}\}$ on the fibres of $E\otimes C$, let the fibre at $x\in G_{n+m,n}\backslash Z_{2k+1}$ be the subspace of the complex n-plane $x\otimes C$ which is orthogonal to the unitary $(n-2k)$ frame u_x, it is a complex $2k$-dimensional space. Choose the first k vector fields $\{e_1,\cdots,e_{2k}\}$ of the sections of he complex vector bundle $E\otimes C$ to be the local sections of the subbundle E', then he connection ∇^C of the complex vector bundle $E\otimes C$ induces a connection ∇' of the ubbundle E':

$$\nabla' e_\alpha = \sum_{\beta=1}^{2k}\omega_{\alpha\beta}e_\beta,\quad (\alpha = 1,\cdots,2k),$$

he matrix of the connection forms of ∇' is $\omega' = (\omega_{\alpha\beta}), (\alpha,\beta = 1,\cdots,2k)$, the corresponding matrix of the curvature forms is $\Omega' = (\Omega'_{\alpha\beta}), (\alpha,\beta = 1,\cdots,2k)$, where

$$-\Omega'_{\beta\alpha} = d\omega_{\alpha\beta} - \sum_{\gamma=1}^{2k}\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta} = \sum_{\lambda=2k+1}^{n+m}\omega_{\alpha\lambda}\wedge\omega_{\lambda\beta}.$$

The kth Chern characteristic form of the complex vector bundle E' is

$$C_{2k}(\Omega') = \frac{1}{(2\pi i)^{2k}\cdot 2k!}\sum_{\alpha,\beta=1}^{2k}\epsilon_{\alpha_1,\cdots,\alpha_{2k},\beta_1,\cdots,\beta_{2k}}\Omega'_{\alpha_1\beta_1}\wedge\cdots\wedge\Omega'_{\alpha_{2k}\beta_{2k}}.$$

Note that the fibres of the subbundle E' are complex $2k$-dimensional spaces, the kth Chern haracteritic form $C_{2k}(\Omega')$ is in fact the Euler characteristic form of the complex vector oundle E'. Using the Chern- Weil technique, we obtain

Lemma 2 *On $G_{n+m,n}\backslash Z_{2k+1}$, there exists a $(4k-1)$-form $\tilde{\tau}$ and a section u of the unitary $(n-2k)$-frame bundle E associated to the complex vector bundle $E\otimes C$ such that*

$$C_{2k}(\Omega^C)-C_{2k}(\Omega')=d(u^*\tilde{\tau}).$$

Proof Constructs the homotopy between the connections ∇^C and ∇'. Let

$$\omega_t=\omega'+tu, u=\omega^C-\omega', \Omega_t=d\omega_t-\omega_t\wedge\omega_t,$$

then $\tilde{\tau}=2k\int_0^1 C_{2k}(u,\Omega_t \underbrace{\cdots}_{(2k-1)},\Omega_t)dt$.

Since $C_{2k}(\Omega')$ is the Euler characteristic form of the complex vector bundle E', according to the results of Chern [1], we have

Lemma 3 *On $G_{n+m,n}\backslash Z_{2k+1}$, there exists a $(4k-1)$- form τ' and a section $\tilde{u}$ such that:*
1) $C_{2k}(\Omega')=d(\tilde{u}^*\tau')$,
2) $\int_{S^{4k-1}} j^*\tau'=-1$,

j is a mapping which identifies the unit sphere S^{4k-1} in the complex $4k$-dimensional Hermitian space C^{2k} to the unit sphere on the fibre of the complex vector bundle E'.

4. The integral formula of the Pontrjagin chracteristic form

Give a $4k$-dimensional chain c_{4k} of $G_{n+m,n}$, we want to calculate the intersection number $c_{4k}\circ Z_{2k}$. Endows a small deformation to c_{4k} such that: 1) $c_{4k}\cap Z_{2k+1}=\emptyset$; 2) c_{4k} intersects transversely with Z_{2k} at finite points $\{a_1,\cdots,a_N\}$, then the unitary $(n-2k)$-frame field u is well-defined on c_{4k}, and the unit vector field $\tilde{u}$ orthogonal to u is defined on $c_{4k}\backslash\{a_1,\cdots,a_N\}$. Decompose the chain c_{4k} more fine such that every simplex of it involves at most one of the zero point of u. Remove a small ball in the simplex involving the zero point a_i with the center a_i and radius ϵ, we denote it by $B_i(\epsilon)$, its boundary is a $(4k-1)$-dimensional sphere $S_i(\epsilon)$. The unit vector field in $\tilde{u}$ is well-defined on $S_i(\epsilon)$. Let $\epsilon\to 0$, the vector field $\tilde{u}$ becomes rotating about the zero point a_i, the rotate number is defined to be the index of the unit vector field $\tilde{u}$ at the zero point a_i, and denoted by $I_{\tilde{u}}(a_i)$, then we have $\lim_{\epsilon\to 0}\tilde{u}(S_i(\epsilon))=I_{\tilde{u}}(a_i)\cdot jS^{4k-1}$, and $c_{4k}\circ Z_{2k}=\sum_{i=1}^N I_{\tilde{u}}(a_i)$. According to the lemma 3,2),

$$\begin{aligned} c_{4k}\circ Z_{2k} &= \sum_{i=1}^N I_{\tilde{u}}(a_i)=-\sum_{i=1}^N I_{\tilde{u}}(a_i)\cdot\int_{S^{4k-1}} j^*\tau'=-\sum_{i=1}^N\int_{I_{\tilde{u}}(a_i)\cdot jS^{4k-1}}\tau' \\ &= -\sum_{i=1}^N\lim_{\epsilon\to 0}\int_{\tilde{u}(S_i(\epsilon))}\tau'=\sum_{i=1}^N\lim_{\epsilon\to 0}\int_{S_i(\epsilon)}\tilde{u}^*\tau'. \end{aligned}$$

Theorem

$$c_{4k}\circ Z_{2k}=(-1)^k\int_{c_{4k}}P_k(\Omega)-\int_{\partial c_{4k}}u^*\tilde{\tau}-\int_{\partial c_{4k}}\tilde{u}^*\tau,$$

where the $(4k-1)$-form $\tilde{\tau}$ and τ' are given by Lemma 2 and 3 respectively.

Proof According to the Lemma 2 and Lemma 3.1),

$$\begin{aligned}(-1)^k\int_{c_{4k}}P_k(\Omega) &= \int_{c_{4k}}C_{2k}(\Omega^C)=\int_{c_{4k}}C_{2k}(\Omega')+\int_{c_{4k}}d(u^*\tilde{\tau})\\ &= \lim_{\epsilon\to 0}\int_{c_{4k}\setminus\bigcup_{i=1}^N B_i(\epsilon)}d(\tilde{u}^*\tau')+\int_{\partial c_{4k}}u^*\tilde{\tau}\\ &= \int_{\partial c_{4k}}\tilde{u}^*\tau'-\sum_{i=1}^{N}\lim_{\epsilon\to 0}\int_{S_i(\epsilon)}\tilde{u}^*\tau'+\int_{\partial c_{4k}}u^*\tilde{\tau}\\ &= \int_{\partial c_{4k}}\tilde{u}^*\tau'+c_{4k}\circ Z_{2k}+\int_{\partial c_{4k}}u^*\tau',\end{aligned}$$

the theorem is proved.

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Grassmann 流形上Pontrjagin 示性式的积分公式

梅向明

(首都师范大学数学系，北京100037)

摘　要

命$G_{n+m,n}$是一个Grassmann 流形，$Z_{2k}=[\underbrace{m-2,\cdots,m}_{2k}-2,m,\cdots,m]$是$G_{n+m,n}$的一个Schubert 流形．命$E$是$G_{n+m,n}$上的规范矢丛，$E\otimes C$是$E$的变化，$\tilde{E}$是$E\otimes C$的相配西$(n-2k)$- 标架丛，并且$E'$是$E\otimes C$的一个子矢丛，本文证明了下列积分公式:

$$c_{4k}\cdot Z_{2k}=(-1)^k\int_{c_{4k}}P_k(\Omega)-\int_{\partial c_{4k}}u^*\tilde{\tau}-\int_{\partial c_{4k}}u^*\tau',$$

其中c_{4k}是$G_{n+m,n}$的$4k$链，$P_k(\Omega)$是$G_{n+m,n}$的第k个Pontrjagin 示性式，$\tilde{\tau}$和τ'是定义在E和E'上的$(4k-1)$- 形式.

关键词 Grassmann 流形，Schubert 流形，Pontrjagin 示性式