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Osciliation Criteria of Solutions for a Class of Boundary Value Problems *

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Abstract In this paper we consider a class of boundary value problems of functional partial differential equations of the neutral type. The principal tool is an averaging technique which enables one to establish such oscillation criteria in terms of related functional differential inequalities.

Keywords Functional partial differential eqations, boundary value problems, oscillation criteria.

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1. Introduction

Recently the oscillation theory for functional differential equations and partial differential equations have undergone an intensive development. However, only a few results have been published so far which deal with the oscillatory properties of the solutions of functional partial differential equations. In the paper of Kreith et al. [1] sufficient conditions were given for oscillation of the solutions of nonlinear hyperbolic differential equations of the form

$$\frac{\partial^2 u}{\partial t^2} - \triangle u + C(x,t,u) = f(x,t),$$

considered in a cylindrical domain.

The aim of the present paper is to generalize results of the paper [1]. In this work sufficient conditions are obtained for oscillation of the solutions of the hyperbolic differential equation of neutral type

$$\frac{\partial^2}{\partial t^2}[u(x,t) + pu(x,t-\tau)] - \triangle u + C(x,t,u) = f(x,t), \tag{1}$$

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where p and τ are positive constants.

2. Main Results

Consider the following problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}[u(x,t)+pu(x,t-\tau)] - \triangle u + C(x,t,u) = f(x,t), & (x,t)\in G \\ \dfrac{\partial u}{\partial n} = g(x,t), & (x,t)\in \sigma \end{cases}, \tag{2}$$

where p and τ are positive constants, G is a cylinder in the space (x,t), i.e., $G \doteq D\times(0,\infty)$, D is a bounded domain in R^n with smooth boundary ∂D. σ is the surface of G, i.e., $\sigma = \partial D\times(0,\infty)$ and n is the vector of the exterior normal to σ. We denote by G_α, $\alpha\in R$ the infinite cylinder $D\times(\alpha,\infty)$.

Suppose that the following conditions (A) hold:

(A_1) $C(x,t,u)\in C(G\times R,R)$, $f(x,t)\in C(G,R)$, $g(x,t)\in C(\sigma,R)$,

(A_2) $C(x,t,\xi)\ge q(t)\varphi(\xi)$ for $(x,t,\xi)\in G\times(0,\infty)$ where $q(t)$ is a continuous and positive function in the interval $(0,\infty)$ and $\varphi(\xi)$ is a continuous, positive and convex function in the same interval $(0,\infty)$.

(A_3) $C(x,t,-\xi) = -C(x,t,\xi)$ for $(x,t,\xi)\in G\times(0,\infty)$.

Corresponding to each solution $u(x,t)$ of the problem (2), we consider the function

$$y(t) = \frac{1}{|D|}\int_D u(x,t)dx, \quad t\in(-\tau,\infty),$$

where $|D| = \displaystyle\int_D dx$.

Lemma 1 *Suppose that (A_1) and (A_2) hold and that $u(x,t)$ is a positive solution of the problem (2) on G_α, $\alpha\ge 0$. Then the function $y(t)$ satisfies the following neutral differential inequality*

$$\frac{d^2}{dt^2}[y(t)+py(t-\tau)] + q(t)\varphi(y) \le G(t)+F(t), \quad t>\alpha, \tag{3}$$

where

$$F(t) = \frac{1}{|D|}\int_D f(x,t)dx, \quad G(t) = \frac{1}{|D|}\int_{\partial D} g(x,t)d\sigma.$$

Proof Let $t>\alpha$ Integrating (1) over D we obtain

$$\frac{d^2}{dt^2}[y(t)+py(t-\tau)] = \frac{1}{|D|}\int_D[\triangle u - C(x,t,u)+f(x,t)]dx, \tag{4}$$

Green's formula yields

$$\int_D \triangle u dx = \int_{\partial D}\frac{\partial u}{\partial n}d\sigma = \int_{\partial D} g(x,t)d\sigma. \tag{5}$$

From (A_2) and Jensen's inequality we get

$$\begin{aligned} \frac{1}{|D|}\int_D C(x,t,u)dx &\geq \frac{1}{|D|}\int_D q(t)\varphi(u)dx \\ &\geq q(t)\varphi\left(\frac{1}{|D|}\int_D u(x,t)dx\right) = q(t)\varphi(y). \end{aligned} \tag{6}$$

Then from (4)–(6) it follows that for $t > \alpha$

$$\frac{d^2}{dt^2}[y(t)+py(t-\tau)] \leq \frac{1}{|D|}\int_{\partial D} g(x,t)d\sigma - p(t)\varphi(y) + \frac{1}{|D|}\int_D f(x,t)dx.$$

This Proves the lemma.

Definition 1 *The solution $u(x,t)$ of the problem (2) is called oscillatory in G if $u(x,t)$ has a zero in G_α for each $\alpha \geq 0$.*

Definition 2[2] *The inequality (3) is called oscillatory at $t=\infty$ if it does not possess a solution which is positive in the interval $[\alpha,\infty)$ for every $\alpha \geq 0$.*

Using Lemma 1. we prove the following theorem.

Theorem 1 *Suppose that the condition (A) hold and the neutral differential inequalities*

$$\frac{d^2}{dt^2}[y(t)+py(t-\tau)] + q(t)\varphi(y) \leq G(t)+F(t), \tag{7}$$

$$\frac{d^2}{dt^2}[y(t)+py(t-\tau)] + q(t)\varphi(y) \leq -G(t)-F(t) \tag{8}$$

are oscillatory at $t=\infty$. Then every solution $u(x,t)$ of the problem (2) is oscillatory in G.

Proof Assume to the contrary that there exists a solution $u(x,t)$ of the problem (2) which has no zero in G_α. If $u(x,t)>0$ in G_α, then from Lemma 1 it follows that $y(t)$ is a positive solution of the inequality (7) for $t>\alpha$, which contradicts the condition of the theorem.

If $u(x,t)<0$ in G_α, then $v(x,t) = -u(x,t)$ is a positive solution of the problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}[v(x,t)+pv(x,t-\tau)] - \triangle v + C(x,t,v) = -f(x,t), & (x,t)\in G \\ \dfrac{\partial v}{\partial n} = -g(x,t), & (x,t)\in\sigma. \end{cases}$$

From Lemma 1 it follows that

$$z(t) = \frac{1}{|D|}\int_D v(x,t)dx, \qquad t>\alpha$$

is a positive solution of the inequality (8) which contradicts the conditions of the theorem as well. This completes the proof of Theorem 1.

The above arguments imply that the oscillation properties of the solutions of the problem (2) can be transformed to investigation of the oscillation properties of the solutions of neutral differential inequalities

$$\frac{d^2}{dt^2}[w(t)+pw(t-\tau)]+f(t,w)\le q(t), \tag{9}$$

where p and τ are positive constants.

Suppose the following conditions (B) hold:

(B_1) $f(t,w)\in C([a,\infty)\times(0,\infty),R)$, $q(t)\in C([a,\infty),R)$, where $a\in R$.

(B_2) $f(t,w)>0$ for $(t,w)\in[a,\infty)\times(0,\infty)$.

Then the following lemma can be proved.

Lemma 2 *Suppose that the conditions (B) hold and*

$$\liminf_{t\to\infty}\frac{1}{t-T}\int_T^t(t-s)q(s)ds=-\infty \tag{10}$$

for every $T>a$. Then the inequality (9) is oscillatory for $t=\infty$.

Proof Assume to the contrary that $w(t)$ is a positive solution of the inequality (9) defined in the interval $[T,\infty)$. Then using the condition (B_2) we have

$$\frac{d^2}{dt^2}[w(t)+pw(t-\tau)]\le q(t). \tag{11}$$

Integrating the above inequality twice in the segment $[T,t]$, we get

$$w(t)+pw(t-\tau)\le C_1+C_2(t-T)+\int_T^t\int_T^r q(s)dsdr.$$

where C_1 and C_2 are constants. Since

$$\int_T^t\int_T^r q(s)dsdr=\int_T^t(t-s)q(s)ds,$$

dividing both sides of the last inequality by $t-T$, we obtain

$$\frac{w(t)+pw(t-\tau)}{t-T}\le\frac{C_1}{t-T}+C_2+\frac{1}{t-T}\int_T^t(t-s)q(s)ds. \tag{12}$$

Let $t\to\infty$, from (10) and (12) it follows that

$$\liminf_{t\to\infty}\frac{w(t)+pw(t-\tau)}{t-T}=-\infty,$$

which contradicts the assumption that $w(t)$ is positive for $t\ge T$. This proves Lemma 2.

Theorem 2 *Suppose that the condition (A) hold and*

$$\liminf_{t\to\infty}\int_T^t(1-\frac{s}{t})(G(s)+F(s))ds=-\infty, \tag{13}$$

$$\limsup_{t\to\infty}\int_T^t(1-\frac{s}{t})(G(s)+F(s))ds=+\infty, \tag{14}$$

for every sufficiently large number T. Then every solution $u(x,t)$ of the problem (2) is oscillatory in G.

Proof From the conditions (13) and (14) it follows respectively that

$$\liminf_{t\to\infty} \frac{1}{t-T}\int_T^t (t-s)(G(s)+F(s))ds = -\infty, \tag{15}$$

$$\liminf_{t\to\infty} \frac{1}{t-T}\int_T^t (t-s)(-G(s)-F(s))ds = -\infty. \tag{16}$$

According to Lemma 2 we obtain that the inequalities (7) and (8) are oscillatory for $t=\infty$. Hence, from Theorem 1 every solution $u(x,t)$ of the problem (2) is oscillatory in G. This completes the proof of the theorem.

Remark Let $p=0$, then our Theorem 1 and Theorem 2 reduce to the Theorem 3.2 and Theorem 3.3 in [1], respectively.

Example Consider the problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}[u(x,t)+u(x,t-\pi)] - \dfrac{\partial^2 u}{\partial t^2} + u(x,t) \\ \qquad = 2e^t\cos x(\sin t+\cos t - e^{-\pi}\cos t), & (x,t)\in G=(0,\frac{\pi}{2})\times(0,\infty), \\ \dfrac{\partial u(0,t)}{\partial x} = 0, & t\in(0,\infty), \\ \dfrac{\partial u(\frac{\pi}{2},t)}{\partial x} = -e^t\sin t, & t\in(0,\infty). \end{cases} \tag{17}$$

One can easily and immediately check that the function

$$\begin{aligned} &C(x,t,u)=u, && (x,t,u)\in G\times R, \\ &f(x,t)=2e^t\cos x(\sin t+\cos t-e^{-\pi}\cos t), && (x,t)\in G, \\ &g(0,t)=0, && t\in(0,\infty), \\ &g(\frac{\pi}{2},t) = -e^t\sin t, && t\in(0,\infty), \end{aligned}$$

satisfy the condition (A). Moreover, we have

$$\begin{aligned} &G(t) = -\frac{2}{\pi}e^t\sin t, && t\in(0,\infty), \\ &F(t) = \frac{4}{\pi}e^t(\sin t+\cos t-e^{-\pi}\cos t), && t\in(0,\infty), \end{aligned}$$

which implies that

$$I(t) = \int_T^t (1-\frac{s}{t})(G(s)+F(s))ds = \frac{e^t}{\pi t}(2\sin t - 2e^{-\pi}\sin t - \cos t) + c,$$

where c does not depend on t. Hence, we have

$$\lim_{t\to\infty}\inf I(t) = -\infty, \tag{18}$$

and

$$\lim_{t\to\infty}\sup I(t) = +\infty, \tag{19}$$

i.e., the conditions (13) and (14) of Theorem 2 hold. It follows from Theorem 2 that every solution of the problem (17) is oscillatory in the cylinder $G(0,\frac{\pi}{2})\times(0,\infty)$. For example,

$$u(x,t) = e^t \sin t \cos x$$

is one such solution.

References

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一类边值问题的振动准则

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摘 要

本文考虑一类中立型泛函偏微分方程的边值问题. 我们的主要工具是平均技巧，利用有关的泛函微分不等式来建立这类边值问题解的振动准则.

关键词 泛函偏微分方程，边值问题，振动准则，平均技巧.