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On Multivariable Blending Approximation *

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Abstract In this paper the generalized Boolean sum of linear operators in higher space are defined, and some approximation properties and applications are given.

Keywords blending function space, generalized Boolean sum, multivariable blending approximation.

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1. Introduction

Let P_k be the space of all polynomials of degree k or less, $B(M)$ the space or bounded and real-valued functions on the interval [0,1]. It is well- known that, the function

$$P:[0,1]^2 \ni (x,y) \longmapsto \sum_{i=0}^{m} x^i A_i(y) + \sum_{j=0}^{n} B_j(x) y^j \in R$$

is called a blending function or a pseudopolynomial, where $A_i, B_j \in B(I)$, and the set of all pseudopolynomials is called the bivariate pseudopolynomial space or blending function space.

The pseudopolynomial space has been introduced by A. Marchhand ([4],[5]), and has later been studied by several authors. For more complete historical information see [6] or [2]. Recently, the degree of approximation from certain bivariate blending function space has been given by H.H.Gonska [2], and some inverse theorems have been established by C.Cottin [3]. In this paper, we shall generalize these results in higher space.

Let us begin with some notions. As usual, R^s will denote the Euclidean space of dimension s, Z_+^s is the set of all nonnegative multi-integers. If $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_s) \in Z_+^s$, then the length and factorial of α are defined by $|\alpha| = \sum_{i=1}^{n} \alpha_i$ and $\alpha! = \alpha_1! \alpha_2! \cdots a_s!$ respectively: $e_i \in Z_+^s$ is an s-dimension vector whose component are zero except that the $i-th$ one is $1 (i = 1, 2, \cdots, s)$, D_y is the differential operator defined by

$$D_y f := \lim_{t \to 0} (f - f(\cdot - ty))/t$$

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and the operator D_{e^i} will be abbreviated to D_i.

For $x \in R^s, \alpha \in Z_+^s$ we define $\|x\| = |x_1| + |x_2| + \cdots + |x_s|, x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_2^{\alpha_s}$ and similary, $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_2^{\alpha_s}$.

Now suppose that there are r Euclidean spaces $R^{s_1}, R^{s_2}, \cdots, R^{s_r}$, and R^s is their tensor product space: $R^s = R^{s_1} \oplus R^{s_2} \oplus \cdots \oplus R^{s_r}, s = s_1 + s_2 + \cdots + s_r$. Let $x \in R^s, x^j$ be the orthogonal projection of x in R_i^s, obviously, if $x^j = (x_1^j, x_2^j, \cdots, x_{s_j}^j)$, then we have $x = (x^1, x^2, \cdots, x^r)$, and $\|x\| = \|x^1\| + \|x^2\| + \cdots + \|x^r\|$. Similar notations can be introduced for the sets of multi-integers $Z_+^{s_1}, Z_+^{s_2}, \cdots, Z_+^{s_r}$.

Let $I = I_1 \oplus I_2 \oplus \cdots \oplus I_r$, where $I_j \subset R^{s_j}$ is non-empty compact set, and $f \in C(I)$. If we fixed $x^i (i \neq j)$ then f becomes a function of x^j on I_j with parametric variables $x^i (i \neq j)$, this function will be denoted by $f_{x_j}(x)$. Furthermore, let L_i be linear operators on $C(I_i), i = 1, 2, \cdots, r$; for $f \in C(I)$, we suppose that

$$L_i := L_i(f_{x_i}(x)).$$

Now we let

$$L_1 \oplus L_2 \oplus \cdots \oplus L_r = \sum_{j=1}^{r} (-1)^{j-1} \sum_{1 \le i_1 < i_2 < \cdots < i_j \le m} L_{i_1} L_{i_2} \cdots L_{i_j}, \tag{1}$$

and $L_1 \oplus L_2 \oplus \cdots \oplus L_r$ called a generalized Boolean sum of $L_1, L_2, \cdots, L_r$. In this paper we will study some approximation properties of operator (1).

2. Error estimation of generalized Boolean operator

In this section we will discuss the approximation order of generalized Boolean operator (1). Before developing our discussion, we need to introduce more notations.

Let $x, h \in R^s, p \in Z^s, f$ is a bounded function in R^s, then the p^{th} order difference operator with step width h at point x is defined as follows:

$$\Delta_{x,h}^p f := \sum_{0 \le l \le p} (-1)^{|p|-|l|} \binom{p}{l} f(x + l \cdot h),$$

where $\binom{p}{l} = \binom{p_1}{l_1} \binom{p_2}{l_2} \cdots \binom{p_s}{l_s}$.

For $f \in C(I), I \subset R^s$ is a nonempty convex compact set, then the $p-th$ order moduli of smoothness is defined by

$$\omega_p(f; \delta) = \sup[\|\Delta_{x,h}^p f\| : x + l \cdot h \in I, \|h\| < \delta, \delta > 0].$$

Let $f \in C(I), I = I_1 \otimes I_2 \otimes \cdots \otimes I_r$, and $x_i (i \neq j)$ are fixed in I_i, then we consider the moduli of smoothness

$$\omega_{p_j}^{(j)}(f, \delta_j) = \sup[\|\Delta_{x^j, h^j}^{p^j} f(x)\| : x^j + l^j \cdot h^j \in I_j, \|h^j\| \neq \delta_j, \delta_j > 0].$$

Obviously the moduli of smoothness defined as above is a mapping from sapce $C(I)$ to $C(\prod_{i \neq j} I_i)$.

Then we have

Lemma 1 *Let*

$$R^s = R^{s_1} \otimes R^{s_2} \otimes \cdots \otimes R^{s_r}.$$

and $f(x) = f(x^1, x^2, \cdots, x^r)$ *is a bounded function in* R^s*, where* $x^j \in R^{s_j}$*, Then* $\forall p = (p^1, p^2, \cdots, p^r), p^j = (p_1^j, p_2^j, \cdots, p_{s_j}^j) \in Z_+^{s_j}$*, we have*

$$\Delta_{x,h}^p f = \Delta_{x^1,h^1}^{p^1} \Delta_{x^2,h^2}^{p^2} \cdots \Delta_{x^r,h^r}^{p^r} f, \tag{2}$$

where

$$h = (h_1^1, h_2^1, \cdots, h_{s_1}^1; h_1^2, h_2^2, \cdots, h_{s_2}^2; \cdots; h_1^r, h_2^r, \cdots, h_{s_r}^r) \in R^s.$$

Lemma 2 *Let* $f \in C(I)$*, then*

$$|\omega_{p^1}^{(1)}(\cdot, \delta_1) \circ \omega_{p^2}^{(2)}(\cdot, \delta_2) \circ \cdots \circ \omega_{p^r}^{(r)}(\cdot, \delta_r) f| \le \omega_p(f, \delta), \tag{3}$$

where $p = (p^1, p^2, \cdots, p^r) \in Z^s, \delta = (\delta_1, \delta_2, \cdots, \delta_r) \in R^s, s = s_1 + s_2 + \cdots + s_r$.

Lemma 3 *Suppose that* $f \in C^\alpha(I)$*, where* $\alpha = (\alpha^1, \alpha^2, \cdots, \alpha^r), \alpha^j = (\alpha_1^j, \alpha_2^j, \cdots, \alpha_{s_j}^j)$*;*

$$M_j : C(I_j) \longmapsto C(I_j)$$

are continuous linear operators. Then

(1) $M_j f \in C^{\alpha'}(I')$*, where*

$$\alpha' = (\alpha^1, \cdots, \alpha^{j-1}, \alpha^{j+1}, \cdots, \alpha^r), I' = (I_1, \cdots, I_{j-1}, I_{j+1}, \cdots, I_r);$$

(2)

$$D^{k^{j'}}(M_j f) = M_j(D^{k^{j'}} f), \tag{4}$$

where $0 \le k^{j'} \le \alpha^j, j \ne j'$.

We now have the following result:

Theorem 1 *Suppose that* $I - I_1 \otimes I_2 \otimes \cdots \otimes I_r, f(x) \in C^\alpha(I)$*, and*

$$L_j : C^{\alpha^j}(I_j) \longmapsto C^{\alpha^{j'}}(I_j)$$

are continuous linear operators which satisfy the following conditions: $\forall f \in C^{\alpha^j}(I_j), x^j \in I_j$*,*

$$\|D^{\beta^j}(f_j - L_j f_j)(x_j)\| \le A_j(x^j)\omega_{p^j}(f_j^{(\alpha^j)}, \Lambda_j(x^j)), \quad j = 1, 2, \cdots, r,$$

where $0 \le \beta^j \le \min(\alpha^j, \alpha^{j'}) = \beta^{j^*}$*,* A_j *and* Λ_j *are some nonnegative functions only depend on* P^j, β^j, L^j *respectively. Then for any*

$$\beta := (\beta^1, \beta^2, \cdots, \beta^r) \ge (\beta^{1^*}, \beta^{2^*}, \cdots, \beta^{r^*}) := \beta^{r^*},$$

and $x \in I$, *we have*

$$\|D^{\beta}(f-(L_1 \oplus L_2 \oplus \cdots \oplus L_r)f\| \le (\prod_{j=1}^{r} A_j(x^j))\omega_p(D^{\alpha}f, \Lambda_{K,L}(x)),$$

where $\Lambda_{K,L}(x) = (\Lambda_1(x^1), \Lambda_2(x^2), \cdots, \Lambda_r(x^r))$.

Proof It is observed easily that, $I - L_j$ and $I - L_1 \oplus L_2 \oplus \cdots \oplus L_r$ are also continuous linear operators, where I is the identity operator. Therefore by the lemma 2, we have

$$\begin{aligned}
&\|D^{\beta}(f-(L_1 \oplus L_2 \oplus \cdots \oplus L_r)f)(x)\|_{\infty} = \|D^{\beta}((I-L_1)(I-L_2)\cdots(I-L_r)f)(x)\|_{\infty}\\
&= \|(D^{\beta_1}(I-L_1))(D^{\beta_2}(I-L_2))\cdots(D^{\beta_{r-1}}(I-L_{r-1}))(D^{\beta_r}(I-L_r))f(x)\|_{\infty}\\
&\le A_1(x^1)\omega_{p'}(\cdot,\Lambda_1(x^1)) \circ \{D^{\alpha^1}(D^{\beta^2}(I-L_2)\cdots D^{\beta^r}(I-L_r))f(x)\}\\
&= A_1(x^1)\omega_p(\cdot,\Lambda_1(x^1)) \circ \{D^{\beta^2}(D^{\beta^2}(I-L_2)\cdots D^{\beta^r}(I-L_r)D^{\alpha'}f(x)\}.
\end{aligned}$$

By the definition of $\omega_p(\cdot,\delta)$, exist a $x^{*^1} \in I_1$ and $h^{*^1} \in R^{s_1}$, such that $x^{*^1} + l^j \cdot h^{*^1} \in I_1, \|h^{*^1}\| < \delta$, and

$$\omega_{p^1}(\cdot,\Lambda_1(x)) = \|\Delta^{p^1}_{x^{*1},h^{*1}}(\cdot)\|_{\infty}.$$

Hence,

$$\begin{aligned}
&\|D^{\beta}(f-(L_1 \oplus L_2 \oplus \cdots \oplus L_r)f)(x)\|_{\infty}\\
&\le A_1(x^1)\|\Delta^{p^1}_{x^{*1},h^{*1}}(D^{\beta^2}(I-L_2)\cdots D^{\beta^*}(I-L_r)D^{\alpha^l}f)(x)\|_{\infty}\\
&= A_1(x^1)\|D^{\beta^2}(I-L_2)\cdots D^{\beta^*}(I-L_2)\Delta^{p^1}_{x^{*1},h^{*1}}D^{\alpha^1}(x)\|_{\infty}.
\end{aligned}$$

Repeat this process r times, we know that, exist $x^* = (x^{*^1}, x^{*^2}, \cdots, x^{*^r})$ and $h^* = (h^{*^1}, h^{*^2}, \cdots, h^{*^r})$ satisfy $x^* + l \cdot h^* \in I, \|h^*\| < \Lambda_{L,H}(x)$ and such that

$$\begin{aligned}
\|D^{\beta}(f-(L_1 \oplus L_2 \oplus \cdots \oplus L_r)f)(x)\|_{\infty} &\le A_1(x^1)A_2(x^2)\cdots A_r(x^1)\|\Delta^{p}_{x^*,h^*}D^{\alpha}f\|_{\infty}\\
&\le A_1(x^1)A_2(x^2)\cdots A_r(x^r)\omega_p(\Delta^{\alpha}f, \Lambda_{K,L}(x)).
\end{aligned}$$

The proof of theorem 1 is complete.

Now we suppose that

$$M_j : C^{\alpha_j}(I_j) \longmapsto C^{\alpha'_j}(I_j), \quad (i = 1, 2, \cdots, r)$$

are linear operators, then we consider the operator U as follows

$$U = \sum_{j=1}^{r}(-1)^{j-1} \sum_{1 \le i_1 < i_2 < \cdots < i_j \le r} L_1 \cdots M_{i_1} \cdots M_{i_2} \cdots M_{i_j} \cdots L_r.$$

Obviously, U is also a linear operator; in addition, if L_i and $M_i (i = 1, 2, \cdots, r)$ are continuous, then U is also continuous.

Theorem 2 *Let*

$$L_j, M_j : C^{\alpha^j}(I_j) \longmapsto C^{\alpha' j'}(I_j) (j = 1, 2, \cdots, r)$$

be linear operators and satisfy the following conditions:

(i) $\forall g\in C^{\alpha}(I_j), x\in I_j$,

$$\|D^{\beta^j}(g-L_jg)(x^j)\|_\infty \le A'_j(x^j)\omega_{p^j}(D^{\alpha^j}g,\Lambda'_{p^j}(x^j)),$$

where $\beta_j\le \alpha^{*j}=\min(\alpha^j,\alpha^{j'})$;

(ii) $\forall g\in C^{\alpha^j}(I_j), x\in I_j$

$$\|D^{\beta^j}(g-M_jg)(x^j)\|_\infty \le A''_j(x^j)\omega_{p^j}(D^{\alpha^j}g,\Lambda''_{p^j}(x^j)).$$

Then for any $f\in C^{\alpha}(I)$ *and* $\beta=(\beta^1,\beta^2,\cdots,\beta^r), x\in I$,

$$\begin{aligned}\|D^{\beta}(f-Uf)(x)\|_\infty \le & \sum_{j=1}^{r}\sum_{1\le i_1<i_2<\cdots<i_j\le r} A''_{i_1}A''_{i_2}\cdots A''_{i_j}\omega_p(D^{\beta(i_1,i_2,\cdots,i_j)}f,\Lambda^{*}_{i_1,i_2,\cdots,i_j}(x))\\ & +\sum_{j=1}^{r}\sum_{1\le i_1<i_2<\cdots<i_j\le r} A'''_1A'''_2\cdots A'''_r\omega_p(D^{\alpha}f,\Lambda^{**}_{i_1,i_2,\cdots,i_j}(x)).\end{aligned}\qquad(5)$$

where

$$A'''_l=\begin{cases}A''_l & \text{if } l\in(i_1,i_2,\cdots,i_j);\\ A'_l & \text{otherwise.}\end{cases}$$

$$\Lambda^{*}_{i_1,i_2,\cdots,i_j}(x)=bigl(\Lambda^{*}_1(x),\Lambda^{*}_2(x),\cdots,\Lambda^{*}_r(x));$$

$$\Lambda^{*}_l(x)=\begin{cases}A''_{p^l}(x)' & \text{if } l\in(i_1,i_2,\cdots,i_j);\\ 0, & \text{otherwise.}\end{cases}$$

$$\Lambda^{**}_{i_1,i_2,\cdots,i_j}(x)=(\Lambda^{**}_1(x),\Lambda^{**}_2(x),\cdots,\Lambda^{**}_r(x)),$$

$$\Lambda^{**}_l(x)=\begin{cases}A''_{p^l}(x^l)' & \text{if } l\in(i_1,i_2,\cdots,i_j);\\ \Lambda^{1}_{p^l}(x^l), & \text{otherwise.}\end{cases}$$

Before to prove this theorem, We prove a theorem in connection with tensor product of operators L_j:

Theorem 3 *Let* $I=I_1\otimes I_2\otimes\cdots\otimes I_r$,

$$L_j: C^{\alpha^j}(I_j)\longmapsto C^{\alpha'^j}(I_j)$$

are continuous linear operators be given, and let $\forall g\in C^{\alpha'^j}, x^j\in I_j$, *the following inequality holds*

$$\|D^{\beta^j}(g-L_jg)(x^j)\|\le A_j(x^j)\omega_{p^j}(D^{\alpha^j}g,\Lambda_j(x^j)),\quad j=1,2,\cdots,r,\qquad(6)$$

where $\beta^j\le\alpha^{j*}=\min(\alpha^j,\alpha'^j)_j A_j(x^j)$ *and* $\Lambda_j(x^j)$ *are bounded function, then*

$$\begin{aligned}&\|D^{\beta}(f-L_1\cdot L_2\cdots L_rf)(x)\|_\infty\\ &\le\sum_{j=1}^{r}\sum_{i\le i_1<i_2<\cdots<i_j\le r}A_{i_1}A_{i_2}\cdots A_{i_j}\omega_p(D^{\beta(i_1,i_2,\cdots,i_j)}f,\Lambda^{*}_{i_1,i_2,\cdots,i_j}),\end{aligned}\qquad(7)$$

where

$$\beta(i_1,i_2,\cdots,i_j)=(\beta^1,\cdots,\beta^{i_1-1},\alpha^{i_1},\beta^{i_1+1},\cdots,\beta^{i_j-1},\alpha^{i_j},\beta^{i_j+1}\cdots,\beta^r),$$

$$\Lambda^*(i_1,i_2,\cdots,i_j)=(0,\cdots,0,\Lambda_{i_1}(x^{i_1}),0,\cdots,0,\Lambda_{i_j}(x^{i_j}),0,\cdots,0).$$

Proof Because

$$\sum_{i+1}^{r}(-1)^{j-1}\sum_{1\le i_1<i_2<\cdots<i_j\le r} I=\sum_{j-1}^{r}(-1)^{j-1}C_r^j\cdot I=I, \tag{8}$$

then the following identity is true:

$$L_1L_2\cdots L_r=\sum_{j=1}^{r}(-1)^{j-1}\sum_{\lambda\le i_1<i_2<\cdots i_j\le r} L_{i_1}\oplus L_{i_2}\oplus\cdots L_{i_j}.$$

Now we have

$$\begin{aligned}
&\|D^\beta(f-L_j\cdot L_2\cdots L-f)(x)\|_\infty\\
&=\|D^\beta\sum_{j=1}^{r}(-1)^{j-1}\sum_{1\le i_1<i_2<\cdots<i_j\le r}(I-L_{i_1}\oplus L_{i_2}\oplus\cdots\oplus L_{i_j})f(x)\|_\infty\\
&=\|\sum_{j=1}^{r}(-1)^{j-1}\sum_{1\le i_1<i_2<\cdots<i_j\le r}D^{\beta'}(I-L_{i_1}\oplus L_{i_2}\oplus\cdots\oplus L_{i_j})D^{\beta''}f(x)\|_\infty\\
&\le\sum_{j=1}^{r}\sum_{1\le i_1<i_2<\cdots<i_j\le r}A_{i_1}(x^{i_1})A_{i_2}(x^{i_2})\cdots A_{i_j}(x^{i_j})\omega_{p^{i_1}p^{i_2}\cdots p^{i_j}}(D^{\beta(i_1,i_2,\cdots,i_j)}f,\Lambda^*)\\
&\le\sum_{j=1}^{r}\sum_{1\le i_1<i_2<\cdots<i_j\le r}A_{i_1}A_{i_2}\cdots A_{i_j}\omega_p(D^{\beta(i_1,i_2,\cdots,i_j)}f;\Lambda_{i_1i_2\cdots i_j}(x)),
\end{aligned}$$

there $\beta'=(0,\cdots,0,\beta^{i'},0,\cdots,0,\beta^{i''},0,\cdots,0),\beta''=\beta-\beta'$. In theorem 3, If $\beta^j=\alpha^j=0$, then (7) will become

$$\|(f-L_1\cdot L_2\cdots L_nf)(x)\|_\infty\le(1-(1-A^*)^r)\omega_p(f,\Lambda(x)), \tag{9}$$

where $A^*(x)=\max(A_1(x^1),A_2(x^2),\cdots,A_r(x^r)),\Lambda(x)=(\Lambda_1(x^1),\Lambda_2(x^2),\cdots,\Lambda_r(x^r))$.

Now we prove the theorem 2. Use again the identity (8),

$$\begin{aligned}
U&=M_1\cdot M_2\cdots M_r-(M_1-L_1)\cdot(M_2-L_2)\cdot(M_r-L_r)\\
&=M_1\cdot M_2\cdot M_r-(I-L\oplus\cdots\oplus L_r)\\
&\quad+\sum_{j=1}^{r}(-1)^{j-1}\sum_{1\le i_1<\cdots i_j\le r}(I-L_1\oplus\cdots\oplus M_{i_1}\oplus\cdots\oplus M_{i_j}\oplus\cdots\oplus L_r)\cdot
\end{aligned}$$

Therefore by Theorem 1 and 3.

$$\begin{aligned}
&\|D^{\beta}(f-Uf)(x)\| \le \|D^{\beta}((I-M_1\cdot M_2\cdots M_r)f)(x)\| \\
&\quad +\sum_{j=1}\sum_{1\le i_1<\cdots i_j\le r}\|D^{\beta}((I-L_1\oplus\cdots\oplus M_{i_1}\oplus\cdots\oplus M_{i_j}\oplus\cdots\oplus L_r)f(x)\| \\
&\le \sum_{j=1}^{r}\sum_{1\le i_1<i_2<\cdots<i_j\le r} A''_{i_1}A''_{i_2}\cdots A''_{i_j}\omega_p(D^{\beta(i_1,i_2,\cdots,i_j)}f,\Lambda^{*}_{i_1,i_2,\cdots,i_j}) \\
&\quad +\sum_{j=1}^{r}\sum_{1\le i_1<i_2<\cdots<i_j\le r} A'''_1A'''_2\cdots A'''_r\omega_p(D^{\alpha}f,\Lambda^{**}_{r,L,M}(x)),
\end{aligned}$$

where

$$A'''_l=\begin{cases} A''_l & \text{if } l\in(i_1,i_2,\cdots,i_j); \\ A'_l & \text{otherwise}, \end{cases}$$

$$\Lambda^{**}_{r,L,M}(x)=(\Lambda'''_1,\Lambda'''_2,\cdots,\Lambda'''_r),$$

$$\Lambda'''_l(x^l)=\begin{cases} \Lambda''_l(x^l), & \text{if } l\in(i_1,i_2,\cdots,i_j); \\ \Lambda'_l & \text{otherwise}. \end{cases}$$

3. Inverse Theorem

The inverse problem for univariate Blending-type approximation has been discussed by C.Cottin [3], Now we discuss the higher-dimension case.

Theorem 4 *Let $L_j^{n_j}: C(I_j)\mapsto C(I_j), n^j\in N^j$ be linear operators, and the following conditions be hold: For all $f\in C(I_j)$ and $x_j\in I_j$,*

$$\|f(x^j)-(L_j^{(n_j)}(f)(x^j)\|_\infty\le\lambda_{n^j}(x^j), n_j\in N$$

imply

$$\omega_{p^j}(f;\delta^j)\le\mu_j(x^j), \tag{10}$$

then for any $f\in C(I)$ and $x\in I$, the condition

$$\|(f-(L_1^{n_1}\oplus L_2^{n_2}\oplus\cdots\oplus L_r^{n_r})f)(x)\|_\infty\le\prod_{j=1}^{r}\lambda_{n_j}(x^j),$$

imply

$$\omega_p(f;\delta)\le\prod_{j=1}^{r}\mu_j(x^j)\,. \tag{11}$$

Proof We assume without loss of generality that λ_{n_j} and $\mu_j(x^j)$ are not equal to zero on I_j. Now let

$$g_r(x^r)=\frac{((I-L_1^{(n_1)})\cdots(I-L_{r-1}^{(n_{r-1})})f)(x_1,\cdots,x_{r-1},x_r)}{\lambda_1(x^1)\lambda_2(x^2)\cdots\lambda_{r_1}(x^{r-1})}.$$

Obviously we have

$$\|(I - L_r^{(n_r)})g_r)(x^r)\| \le \lambda_r(x^r).$$

Then by assumption we can prove that, $\forall h^r \le \delta^r$, there holds

$$|(I - L_1^{(n_1)}) \cdots (I - L_{r-1}^{(n_{r-1})}) \cdot (\frac{\Delta_{x^r,h^r}^{p^r} f(x_1, \cdots, x_{1r}, x_r)}{\mu_r(x^r)})| < \lambda_1(x^1)\lambda_2(x^2) \cdots \lambda_{r-1}(x^{r-1}).$$

Repeat this process r times, we obtained

$$|\Delta_{x,h}^p f(x)| < \mu_1(x^1)\mu_2(x^2) \cdots \mu_r(x^r).$$

Because this inequality holds for all $h < \delta$, therefore we have

$$\omega_p(f, \delta) \le \mu_1(x^1)\mu_2(x^2) \cdots \mu_r(x^r),$$

which is just (11).

4. Application

As applications of theorem 1, we discuss first the piecewise pseudopolynomial function space. Let $I = [0,1], p_j = 1,2,3$ or 4 and

$$S_{\Delta_{n_j}} : C^{p_j}(I) \longmapsto C^2(L) \quad (j = 1,2,3)$$

be the interpolation spline operators with respect to partition $\Delta_{n_j} = \{x_0^j, x_1^j, \cdots, x_n^j\}$ and satisfies condition

$$\|(f - S_{\Delta_j} f)^{k_j}\|_\infty \le C(p_j, k_j) \cdot \delta_j^{p_j - k_j} \omega_{4-p_j}(f^{(p_j)}; \delta_j),$$

where δ_j is the mesh gauge of Δ_{n_j} (cf. [3]). Now we consider the linear operator $U_{\Delta_n} : C^p(I^3) \ni f \mapsto S(I^3, \Delta_n)$ as follows:

$$\begin{aligned} U_{\Delta_n} = \; & S_{x,\Delta_{n_1}} + S_{y,\Delta_{n_2}} + S_{z,\Delta_{n_3}} \\ & -(S_{x,\Delta_{n_1}} \circ S_{y,\Delta_{n_2}} + S_{y,\Delta_{n_2}} \circ S_{z,\Delta_{n_3}} + S_{z,\Delta_{n_3}} \circ S_{x,\Delta_{n_1}}) \\ & + S_{x,\Delta_{n_1}} \circ S_{y,\Delta_{n_2}} \circ S_{z,\Delta_{n_3}}, \end{aligned}$$

where $I^3 = I \times I \times I, \Delta_n = \Delta_{n_1} \times \Delta_{n_2} \times \Delta_{n_3}$. by theorem 1, we have

Theorem 5 *Let $S_{\Delta_{n_j}}(j = 1,2,3)$ and u be given above, then for any $f \in C^p(I^3)$.*

$$\|D^k(f - U_{\Delta_n} f)\|_\infty \le C(p,k)\delta^{|p|-|k|}\omega_{4-p}(D^p f, \delta).$$

where $p = (p_1, p_2, p_3), k = (k_1, k_2, k_3), \delta = (\delta_1, \delta_2, \delta_3), 4 - p = (4 - p_1, 4 - p_2, 4 - p_3)$.

Wang and Chui [8] have introduced the bivariate splines operator $V_{m,n} : C(\Omega) \mapsto S_2^1(\Delta_{n_1,n_2})$ as follows:

$$V_{m,n} = \sum_{i,j} f(x_j, y_j)\beta_{i,j}(x, y),$$

and some error bounds have been obtained. Where $\Omega \supset R = I^2$ is open set, the meaning of Δ_{n_1,n_2} and $B_{i,j}(x,y)$ are referred to [8]. Using the symbol in [8] and note that $3\delta_{n_1,n_2} < \delta'_{n_1,n_2} < \sqrt{10}\delta_{n_1,n_2}$ and $\omega_K(f,\delta'_{n_1,n_2}) \le 3\omega(f,\frac{1}{n_1},\frac{1}{n_2})$, then we can rewrite the results in [8] as follows

$$\|f - V_{n_1,n_2}f\| \le 12\omega(f,\frac{1}{n_1},\frac{1}{n_2}),$$

$$\|f - V_{n_1,n_3}f\| \le 3\delta_{n_1,n_2}\max_{a=1}\omega(D^\alpha f,\frac{1}{2n_1},\frac{1}{2n_2}).$$

Let T_{n_3} be Schoenberg's variation diminishing quadratic spline operator with respect to partition $\Delta_n = \{0,\frac{1}{n_3},\cdots,\frac{n_3-1}{n_3},1\}$:

$$T_{n_3} = \sum_j f(\frac{z_{j+1}+z_{j+2}}{2})N_{n,j}(x),$$

where N_j are normalized B-Spline (cf. [7]), then we have

$$\|(f - T_{n_3},f)(x)\| \le 2\omega(f,\frac{1}{n_3}),\quad \|(f - T_{n_3},f)(x)\| \le \frac{2}{n_3}\omega(f',\frac{1}{n_3}).$$

Therefore, if we let

$$\begin{aligned}(U_nf)(x,y,z) &= (V_{n_1,n_2,x,y} + T_{n_3,z} - Vn_1,n_2,x,y\circ T_{n_3},z)f(x,y,z)\\ &= \sum_{j_1,j_2,j_3\in Z^2} f(x_{j_1},y_{j_2},z)B_{j_1,j_2}(x,y) + \sum_{j_3\in Z} f(x,y,\frac{z_{j_3+1}+z_{j_3+2}}{2})N_{n_3,j}(z)\\ &\quad - \sum_{j_1,j_2,j_3\in Z^3} f(x_{j_1},y_{j_2},\frac{z_{j_3+1}+z_{j_3+2}}{2})B_{j_1j_2}(x,y)N_{j_3}(z),\end{aligned}$$

where $n = (n_1,n_2,n_3)$. Then by the theorem 3 we have

Theorem 6 *Let V_{n_1,n_2} and V_{n_3} be given as above, then for any $f(x,y,z)\in C(I^3)$,*

$$\|(f - U_nf)(x)\|_\infty \le 24\omega(f,\frac{1}{n_1},\frac{1}{n_2},\frac{1}{n_3}),$$

for any $f\in C^1(I^3)$

$$\|(f - U_nf)(x)\|_\infty \le \frac{6}{n_3}\cdot\delta_{n_1,n_2}\cdot\max_{\alpha=1}\omega(D^\alpha f,\frac{1}{2n_1},\frac{1}{2n_2},\frac{1}{n_3}).$$

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多元混合逼近

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摘 要

本文定义了高维空间中任意多个线性算子的广义Boolean 和，并给出了逼近度估计和逆定理及一些应用.