Journal of Mathematical Research & Exposition
Vol. 17, No. 4, 513- 518, Nov. 1997

Power Groups and Order Relations*

Yang Wenze
(Dept. of Math., Yunnan Educational College, Kunming 650031)

Abstract Let G be a non-monoidal group, E be a normal subsemigroup of G such that $E^2 = E$ and $1_G \notin E$. Then we can define a partial order in G by setting E as the positive cone, and G becomes a partially ordered group. Then we can study both the group G and power group Γ on G with identity E by the order. The structure of G is clear if the order is maximal and the power group Γ on G can be expanded to be of quasi-quotient type if G is lattice ordered.

Keywords ordered group, semigroup, power group, monoidal group.

Classification AMS(1991) 06F15, 20M10/CCL O152

1. Introduction

Let G be an arbitrary group, all non-empty subsets of G form a monoid $P(G)$ under the subset multiplication. A subgroup Γ of $P(G)$ is called a power group on G, and G, the generating group of Γ. If N is a normal subgroup of G then G/N is a power group on G, whose elements are cosets of N and multiplication can be done by their representatives, that is $aN \cdot bN = abN$. Does every power group on G behave in this way, or is it of quotient type? The answer is yes if G is a torsion group.[6] But there exists another improtant type of power groups which is not of quotient type: Let E be a non-empty subset of G satisfying $E = E^2 = E$. $E = \{e_1e_2 | e_i \in E\}$, and H, a subgroup of $N_G(E) = \{g \in G | g^{-1}Eg = E\}$, then $\Gamma = \{hE | h \in H\}$ is a power group on G which is not of quotient type if E is not a subgroup of G. For example, replacing G by Q^+, the multiplicative group of positive rational numbers, and E by Z^+, the set of all positive integers, we get a power group $\Gamma = \{qZ^+ | q \in Q^+\}$, in which $q_1Z^+ \cdot q_2Z^+ = q_1q_2Z^+$, but Γ is not of quotient type because its elements are not cosets of any subgroup of Q^+. However, this new type of power groups does not go much further than the quotient one- it "looks like" quotient, so we call it quasi-quotient. Of cause, a power group of quotient type is also of quasi-quotient type. Thus comes another question: Is the type of every power group on G quasi-quotient? The answer is no in general. But we have such a result in [7], that if the identity 1_G of G is contained in E, the identity of power group Γ on G, then Γ is of quasi-quotient type. That makes the group theoretical property P worthwhile to study, where P is that every non-empty subset E of G satisfying $E^2 = E$ contains 1_G. A group G is called monoidal if it has the property P. So the type of every power group on a monoidal group is certainly quasi-quotient. Unfortunately, monoidality is a fairly restrictive property. It has been proved that in a quite

* Received Jun. 25, 1994. Supported by the National Natural Science Foundation of China.

large class of groups, monoidality is equivalent to being torsion-by-cyclic-by-finite. Even the group $Z \oplus Z$ is not monoidal [5]. So there are plenty of groups such that some power groups on which may not be of quasi-quotient type. For example, let $G = Z \oplus Z$ and $A_r = \{(x,y) \mid \sqrt{2}x + y > r,\ x, y \in Z,\ r \in R\}$. Then $\Gamma = \{A_r \mid r \in R\}$ is a power group on G with identity $E = A_0$ not containing $(0,0)$ and group operation $A_{r1} + A_{r2} = A_{r1+r_2}$. This power group Γ on $G = Z \oplus Z$ is not of quasi-quotient type since if r is an irrational number other than $m\sqrt{2} + n\ (m, n \in Z)$, then A_r can not be expressed in the form $A_r = (x_r, y_r) + E$ with $(x_r, y_r) \in Z \oplus Z = G$. But embedding G in $\overline{G} = R \oplus R$, and setting $\overline{A}_r = \{(x,y) \in \overline{G} \mid \sqrt{2}x + y > r,\ r \in R\}$, we get a power group $\overline{\Gamma}$ on $\overline{G}$ where $\overline{\Gamma} = \{\overline{A}_r \mid r \in R\}$ satisfying $A_r = \overline{A}_r \cap G$, $\overline{\Gamma} \cong \Gamma$ and the type of $\overline{\Gamma}$ is quasi-quotient. Generally, let G be a subgroup of $\overline{G}$, and Γ, $\overline{\Gamma}$ be power group on G, $\overline{G}$ respectively, if $\overline{A} \to \overline{A} \cap G$ is an isomorphism from $\overline{\Gamma}$ to Γ, then $\overline{\Gamma}$ is called an expansion of Γ. Can any power group be expanded to be of quasi-quotient type? This is what the paper will deal with.

Let G not be monoidal, E be a non-empty subset of G such that $1_G \notin E$, $E^2 = E$ and $G = N_G(E)$, then G can be partially ordered by defining $P = \{1_G\} \cup E$ as its positive cone. In that case, we can study both groups G and Γ by the order. Adopting the notation in [1], by po-group, fo-group l-group and O-group we mean partially ordered, fully ordered, lattice-ordered and fully orderable group respectively. Let Γ be a power group on G with identity E, then every element A of Γ amm a has a lower bound in G (see Lemma 2). And if every element of Γ has a g. l. b. in G, then Γ is of quasi-quotient type (see Lemma 3). So, if G can be embedded in a complete l-group G, then Γ can be expanded to be of quasi-quotient type. However, an l-group can be embedded in a complete l-group if and only if it is Archimedean, and an Archimedean group must be Abelian.[3] Thus, the l-groups which can be embedded in a complete l-group are very limited. But the completeness turns out to be unnecessary for power group expansion since we have proved in this paper that if G is an l-group, then any power group Γ on G with identity E as the positive cone of G possesses an expansion $\overline{\Gamma}$ which is of quasi-quotient type (Theorem 2). It has also been abtained in this paper that if the identity E is maximal and normal, then the po-group G is dericted, and there exists a normal subgroup N of G such that G/N is an O-group (Theorem 1), which yield immediately a corollary that an abelian group is an O-group if and only if it is torsoin-free.

2 Partial orders and power groups

In this paper, we study only those power groups whose identity does not contain the identity of its generating group.

Let Γ be a power group on G and E, the identity of Γ. Since $1_G \notin E$ and $E^2 = E$, an order $\leq$ in G can be defined by setting its positive cone $P = \{1_G\} \cup E$, i. e., $a \leq b$ if and only if $a^{-1}b \in P$, which is called order E or order Γ. The order is left isotone, i. e., $a \leq b$ implies $ca \leq cb$ for any a, b, c in G. If E is normal in G, then the order E is also right isotone and G becomes a po-group with order E. Since $E^2 = E$, for any $e \in E$ there exists $e' \in E$ such that $1_G < e' < e$, thus order E is dense in G.

Conversely, if G is an order dense po-group and E is the set of all strictly positive elements of G, then $1_G \notin E$ and $E^2 = E$. So, there exists a power group on G with identity E. Upon the fact, whenever talking about power group on an order dense po-group, we always mean this kind of power groups throughout this paper.

Let $\leq_1$ and $\leq_2$ are orders in set G, then order $\leq_2$ is called an extension of order $\leq_1$ if $a \leq_1 b$ always implies $a \leq_2 b$ for any $a, b \in G$. Obviously, any order is its own extension, the trivial extension. An order is maximal if it has only the trivial extension.

Lemma 1 *There exists a maximal order in a non-monoidal group among the orders defined by power groups on it.*

Proof Let G be a non-monoidal group. Denote

$$S = \{X \subseteq G \mid 1_G \notin X,\ X^2 = X\}.$$

By Zorn's Lemma, there exists a maximal element E in S, which is desired. □

Theorem 1 *Let Γ be a power group on G such that the order Γ is maximal. Let $N = \{n \in G \mid nE = En = E\}$, the representative set of the identity E of Γ. If $G = N_G(E)$, then*

(1) *N coincides with the maximal subgroup of G such that $N \cap E = \emptyset$;*

(2) *N is monoidal;*

(3) *G is a dericted group;*

(4) *G/N is an O-group.*

Proof (1) It is obvious that N is a subgroup of G and $N \cap E = \emptyset$. Let K be a subgroup of G satisfying $K \cap E = \emptyset$. Since $(KE)^2 = K^2E^2 = KE$ and $E \subseteq KE$, it follows that $KE = E$ or $1_G \in KE$ by the maximality of E. If $1_G = ke$ with $k \in K$ and $e \in E$, then $k^{-1} \in K \cap E$ contrarying to $K \cap E = \emptyset$. Therefore $KE = E$ and $K \subseteq N$.

(2) If N possesses a non-empty subset E_1 such that $E_1^2 = E_1$ and $1_G \notin E_1$, then $E^* = E \cup E_1$ also has the property, a contradiction.

(3) Let $g \in G$ be a fixed element. Denote

$$T = \cup g^iE \qquad (0 \leq i < +\infty),$$

then $T^2 = T$ and $E \subseteq T$. We claim that $g \in EE^{-1}$ where $E^{-1} = \{e^{-1} \mid e \in E\}$. Suppose it is false, then gE is not a subset of E and $E \neq T$, that forces $1_G \in T$ by the maximality of E. Then there exists a positive integer n such that $1_G \in g^nE$. Since $g^nE = g^nE \cdot E$, so $E \subseteq g^nE$. Similary, there exists a positive integer k such that $E \subseteq g^{-k}E$ for $g^{-1} \notin EE^{-1}$. From $E \subseteq g^nE$, we conclude that $E = E^k \subseteq (g^nE)^k = g^{nk}E$ and from $E \subseteq g^{-k}E$, $E \subseteq g^{-nk}E$, so $g^{nk}E = E$. But $1_G \in g^{nk}E$, that is impossible.

Therefore, $g \in EE^{-1} = E^{-1}E$ for all $g \in G$. Acording to Clifford[2], G is a dericted group.

(4) For any $g \in G$, $n \in N$, $n^gE = g^{-1}ngE = g^{-1}(nE)g = g^{-1}Eg = E$, so $N \lhd G$. And the order induced in N is trivial by (1), so N is convex.

By factoring out N we can assume $N = 1$. For any $g \in G$, either $gE \subseteq E$ or $g^{-1}E \subseteq E$ by the proof of (3), i.e., either $g^{-1} \in L(E)$ or $g \in L(E)$ where $L(E)$ is the set of all lower bounds of E. Taking $L(E)^{-1}$ to be the positive cone, we makes G into an fo-group. □

As a corollary, we can now easily obtain Levi's Theorem [4].

Corollary (Levl) *An abelian group is an O-group if and only if it is torsion-free.*

Proof Let G be a torsion-free abelian group. If G is monoidal, then, by [5], $G \cong Z$, an fo-group. While G is non-monoidal, let N, E be defined as in Theorem 1. Now G/N is an O-group and N is monoidal, therefore $N \cong Z$ or $\{1_G\}$, which result in that G is also an O-group, because the class of O-groups is closed with respect to forming extension. □

Lemma 2 *Let Γ be a power group on G, then every element of Γ has a lower bound in G with respect to the order Γ.*

Proof Let $A \in \Gamma$, and A^{in} be the inverse of A in Γ, then $A^{in}A = E$. Taking an element $a' \in A^{in}$, we get $a'A \subseteq A^{in}A = E$, i.e., $A \subseteq (a')^{-1}E$. Therefore $(a')^{-1} < a$ for all $a \in A$, thus $(A^{in})^{-1} \subseteq L(A)$. □

Lemma 3 *Let Γ be a power group on G, then Γ is of quasi-quotient type if each element of Γ has a g.l.b. in G. This condition is also necessary when G is an l-group with respect to order Γ.*

Proof Let $A \in \Gamma$, and a^* is a g.l.b. of A, then $A \subseteq a^*E$. By Lemma 2, $(A^{in})^{-1} \subseteq L(A)$, so $a^* \geq (a')^{-1}$ for all $a' \in A^{in}$. Therefore $(a^*)^{-1} \in L(A^{in})$ and $(a^*)^{-1}E \supseteq A^{in}$, that implies $a^*E \subseteq A$. Now $a^*E = A$ and a^* is a representative of A.

If G is an l-group and $A = a^*E$, then $a^* \in L(A)$ and $a^* \vee c \in L(A)$ for any $c \in L(A)$. Suppose $a^* \vee c > a^*$, then $a^* \vee c \in a^*E = A$, so $A \cap L(A) \neq \emptyset$. But that is impossible because $A = AE$, which means that there is no minimal element in A. Hence $a^* \vee c = a^*$ and a^* is a g.l.b. of A.

3 Expansion of power groups

By Lemma 3, if G is a complete l-group, then every power group on G is of quasi-quotient type. Therefore, if a po-group can be embedded in a complete l-group, then the power group on it can be expanded to be of quasi-quotient type. But, actually, the completeness is not necessary by the following

Theorem 2 *Let Γ be a power group on G and G be an l-group with respect to order Γ, then Γ expands to a power group of quasi-quotient type.*

Proof Let G be an order-dense l-group. A non-empty subset X of G such that $U(X) \neq \emptyset$ associates with a subset $X^{\#} = LU(X)$, where $U(X)$ and $L(X)$ are the sets of upper and lower bounds in G of X respectively. It is easy to check the following properties:

(1) $X \subseteq X^{\#}$,

(2) $(X^{\#})^{\#} = X^{\#}$,

(3) $X \subseteq Y$ implies $X^{\#} \subseteq Y^{\#}$.

Let

$$M = \{X^{\#} \mid \emptyset \neq X \subseteq G,\ U(X) \neq \emptyset\}.$$

and define a composition $*$ in M as

$$X^{\#} * Y^{\#} = (X^{\#} Y^{\#})^{\#},$$

where $X^{\#} Y^{\#}$ is the set of all xy with $x \in X^{\#}$, $y \in Y^{\#}$. Applying (1)- (3), one can check that M is a complete l-monoid with identity $1_G^{\#} = LU(1_G)$ and set inclusion as its partial order.

Let $\overline{G}$ denote the subgroup of M consisted of all its units and define

$$\Phi: g \to g^{\#} = LU(g)$$

for $g \in G$, then Φ is an o-monomorphism from G to M, and $\overline{G}$ is the Dedekind extension of G[1,3].

Let $E, \overline{E}$ be the sets of all strictly positive elements of $G, \overline{G}$ respectively. Since G is order-dense, $E^2 = E$ and $\Phi(E) * \Phi(E) = \Phi(E)$. Claim $\overline{G}$ is also order-dense, i.e., $\overline{E} * \overline{E} = \overline{E}$. In fact, if $X^{\#} \supset 1_G^{\#}$, there exists $x \in X^{\#}$ such that $x \not\leq 1_G$. Now $1_G^{\#} \subset \{x \vee 1_G\}^{\#} = \{x, 1_G\}^{\#} \subset X^{\#}$ and $\{x \vee 1_G\}^{\#} \in \Phi(E)$, so $\overline{E} = \Phi(E) * \overline{E} \subseteq \overline{E} * \overline{E}$.

Let Γ be a power group on G with identity E, for each $A \in \Gamma$ assign an

$$\overline{A} = \Phi(A) * \overline{E} = \{a^{\#} * X^{\#} \mid a \in A,\ X^{\#} \in \overline{E}\}.$$

Since $\Phi(E) \subseteq \overline{E}$ and $\Phi(A) = \Phi(AE) = \Phi(A) * \Phi(E)$, we get $\Phi(A) = \Phi(G) \cap \overline{A}$. Thus, $A \to \overline{A}$ is an isomorphism from Γ to $\overline{\Gamma} = \{\overline{A} \mid A \in \Gamma\}$, and $\overline{\Gamma}$ is an expansion of Γ.

We need to prove that $\overline{\Gamma}$ is of quasi- quotien type. It is sufficient to show that every element of $\overline{\Gamma}$ has a g. l. b. in $\overline{G}$ by Lemma 3. Because $\overline{A} = \Phi(A) * \overline{E}$, what we really need to show is that $\Phi(A)$ has a g. l. b. in $\overline{G}$ for any $\overline{A} \in \overline{\Gamma}$.

By Lemma 1, $L(A) \neq \emptyset$ so $(L(A))^{\#} \in M$. If $a \in A$, then $a \in UL(A)$, that implies $a^{\#} = L(a) \supseteq LUL(A) = (L(A))^{\#}$, therefore $(L(A))^{\#}$ is a lower bound of $\Phi(A)$ in M. On the other hand, if $X^{\#} \subseteq a^{\#}$ holds for all $a \in A$ and some $X^{\#} \in M$, so does $x \leq a$ for all $x \in X^{\#}$, $a \in A$, hence $X^{\#} \subseteq L(A)$. Now we conclude that $X^{\#} = (X^{\#})^{\#} \subseteq (L(A))^{\#}$, so $L(A))^{\#}$ is a g. l. b. of $\Phi(A)$ in M.

This last step is to show $(L(A))^{\#} \in \overline{G}$. By Lemma 2, $(A^{in})^{-1} \subseteq L(A)$, so $(L(A))^{\#} \bullet (L(A^{in}))^{\#} \supseteq L(A) \bullet L(A^{in}) \supseteq (A^{in})^{-1} \bullet A^{-1} = (A \bullet A^{in})^{-1} = E^{-1}$, therefore $(L(A))^{\#} * (L(A^{in}))^{\#} = ((L(A))^{\#} \bullet (L(A^{in}))^{\#})^{\#} \supseteq (E^{-1})^{\#} = 1_G^{\#}$. On the other hand, $L(A)L(A^{in}) \subseteq L(AA^{in}) = L(E) = 1_G^{\#}$ implies that $L((A))^{\#} * (L(A^{in})^{\#} = (L(A) \bullet L(A^{in}))^{\#} \subseteq (1_G^{\#})^{\#} = 1_G^{\#}$. Thus, $(L(A^{in})^{\#}$ is the inverse of $(L(A))^{\#}$ and $(L(A))^{\#} \in \overline{G}$, the result follows □

Now, every element $\overline{A}$ of $\overline{\Gamma}$ can be expressed in the form $\overline{A} = (L(A))^{\#} * \overline{E}$. Expecially, if $A = aE$, then $(L(A))^{\#} = a^{\#}$ and $\overline{A} = a^{\#} * \overline{E}$.

Acknowledgements The author is grateful to Prof D. J. S. Robinson for his comments

References

[1] G. Birkhoff, *Lattice theory*, Ame. Math. Soc., Providence, Rhode Island, 1973, 287- 319.

[2] A. H. Clifford, *Partially ordered abelian groups*, Annals Math., **41** (1940), 456- 473.

[3] L. Fuchs, *Partially ordered algebraic systems*, Pergamon Press, Oxford London New York Paris, 1963, 9 - 100.

[4] F. W. Levi, *Arithmetische gesetze im gebiete diskreter gruppen*, Rend. Palermon, **35** (1913), 225- 236.

[5] D. J. S. Robinson and W. Yang, *On subsemigroups and submonoids of groups*, Ricerche di Matematica, **43** (1994), 173- 179.

[6] W. Yang, *Groups where every elements is noempty subset of a group*, J. Southwest Teach. Col., **2** (1985), 106- 108.

[7] W. Yang, *Groups in which every element is a non- empty subset of a group*, J. Yunnan Educ. Col., **3** (1988), 1- 4.

幂群与序关系

杨　文　泽

(云南教育学院数学系, 昆明650031)

摘　　要

设 G 为非monoidal群, E 是它的正规子集, 满足 $E^2 = E$ 并且 $1_G \notin E$. 利用 E 作为正锥, 可以在 G 上定义一个偏序, 并且 G 成为一个偏序群. 这样就可以利用这个序关系同时研究群 G 以及 G 上的以 E 为单位元的幂群. 当 E 是极大子半群时, 得到 G 的一个结构定理; 在 G 是格序群的条件下, G 上的幂群 Γ 可以膨胀为一个拟商群.