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The Nonstandard Representation of Optimal Stopping of a Class of Right Continuous Processes on Loeb Spaces *

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Abstract In the paper,we obtain the SRC lifting of a right continuous, left upper semi-continuous random process on a Loeb space.And show the existence of S-optimal stopping of an internal process,construct the S-optimal stopping. Finally we prove the fact that the standard part of the S-optimal stopping of a SRC lifting is the optimal stopping of the corresponding standard process,which generalizes the conclusions in [8] on a Loeb space.

Keywords SRC lifting, standard part, S-optimal stopping.

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1. Introduction

In [8],Thompson showed the existence of the optimal stopping of the random process that is right continuous and quasi-left upper semi-continuous.And Xian[9], Jin[10] solved many related problems.However,it is very troublesome and inconvenient in practice to check whether the conditions pointed by them hold.In the paper,we generalize the conclusions in [8–10] with the assumptions that the random process defined on a Loeb probability space is right continuous and left upper semi-continuous.Obviously our conditions are more applicable and easier to verify.

It is well known that the optimal stopping problem of discrete time was well solved,and the optimal stopping was constructed by the backward induction method. Can we extend the methods and the results in the discrete time case to the continuous time one? Nonstandard probability theory,developed by Anderson [2],Loeb[4],Hoover and Perkins[5],Keisler[6] provides an efficient tool for solving the problem.However,apart from Dalang[14],we have so far found little information for studying the optimal stopping problems via nonstandard analysis method.

The following are the main conclusions of the paper.We give the lifting of a class of right continuous processes,which generalizes the results in Hoover and Perkins[5].And with

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the hyperfinite backward induction method,we construct the optimal stoppings of right continuous and left upper semi-continuous processes via the standard part mappings of the S-optimal stoppings of internal processes. This generalizes the conclusions in [8–10].

Assume we study the problems in an ω_1-saturation enlargement of a superstructure $V(S)$,where S contains the real number set R.And $\mathrm{R}_+ = \{x \in \mathrm{R} | x \geq 0\}, \overline{\mathrm{R}}_+ = \mathrm{R}_+ \bigcup\{\infty\}$,N denotes the set of all natural numbers,$\mathrm{N}_0 = \mathrm{N}\bigcup\{0\}$.$(\Omega, \mathcal{B}, \overline{P})$ is a $*-$finite probability space[2–4],whose Loeb space is $(\Omega, L(\mathcal{B}), L(\overline{P}))$.For convenience we denote $L(\overline{P})$ by $P, L(\mathcal{B})$ by $\mathcal{F}$. $\overline{E}$ denotes the internal expectation with respect to $\overline{P}$.Let $\eta \in {}^*\mathrm{N} \setminus \mathrm{N}$,and $\eta_0 = \eta!, \Delta t = 1/\eta_0$.The $*-$finite set $T = \{k \cdot \Delta t | 0 \leq k \leq \eta_0^2\}$.We may obtain by [5] that T is a S-densely $*-$finite subset of ${}^*\mathrm{R}_+$,i.e., $\{{}^\circ\underline{t} | \underline{t} \in \mathrm{ns}(T)\} = \mathrm{R}_+$,and T contains all nonnegative rational numbers.For the other notions and notations refer to [5].

2. SRC lifting

Corresponding to the right continuous,left upper semi-continuous functions,we introduce the following definition.

Definition 2.1 *Let the internal mapping $F : T \longrightarrow {}^*R$, ${}^\circ|F(\underline{t})| < +\infty$ for all $\underline{t} \in ns(T)$.If for each $t \in R_+, \underline{s}, \underline{t}_1, \underline{t}_2 \in T$ satisfying $\underline{s} \approx \underline{t}_1 \approx \underline{t}_2 \approx t, \underline{s} < t$ and $t \leq \underline{t}_1, t \leq \underline{t}_2$, we have*

$$^\circ F(\underline{s}) \leq {}^\circ F(\underline{t}_1), F(\underline{t}_1) \approx F(\underline{t}_2).$$

F is called the class SRC on T.

If $X : T \times \Omega \longrightarrow {}^*R$ is an internal process,and $X(\cdot, w)$ is the class SRC on T for a.a.$w \in \Omega$,*then X is called the class SRC on T.*

RLL denotes the set of all right continuous,left upper semi-continuous processes on R_+.

Definition 2.2 *Suppose the process $x = \{x_t | t \in R_+\} \in RLL$,the internal process X : $T \times \Omega \longrightarrow {}^*R$.For a.a.$w \in \Omega$,the following*

(i) For each $\underline{t} \in ns(T), \underline{t} \geq {}^\circ\underline{t} \Longrightarrow {}^\circ X(\underline{t}, w) = x({}^\circ\underline{t}, w)$;

(ii) For each $\underline{t} \in ns(T)$, ${}^\circ\underline{t} > \underline{t} \Longrightarrow {}^\circ X(\underline{t}, w) \leq x({}^\circ\underline{t}, w)$

hold.X is called SRC lifting of x.

Obviously,if X is the SRC lifting of the process x, X is the class SRC on T.

Let $\{\mathcal{B}_{\underline{t}} | \underline{t} \in T\}$ be the internal filtration in $(\Omega, \mathcal{B}, \overline{P})$, i.e.the internally nondecreasing collection of internal sub-algebra of $\mathcal{B}$. For each $t \in \mathrm{R}_+$,we write

$$\mathcal{F}_t = (\bigcap_{\underline{t} \in T, {}^\circ\underline{t} > t} \sigma(\mathcal{B}_{\underline{t}})) \vee \mathcal{N},$$

where $\mathcal{N}$ means the class of $P-$null sets in $\mathcal{F}$.We call $\{\mathcal{F}_t | t \in \mathrm{R}_+\}$ the standard part of $\{\mathcal{B}_{\underline{t}} | \underline{t} \in T\}$.It is easy to show that $\{\mathcal{F}_t | t \in \mathrm{R}_+\}$ is right continuously nondecreasing collection of $\sigma-$algebras. For each $t \in \mathrm{R}_+$,let

$$\mathcal{H}_t = \bigcup\{\sigma(L(\mathcal{B}_{\underline{t}}) \cup \mathcal{N}) | \underline{t} \in T, \underline{t} \approx t\}.$$

In [3],it was shown that $\{\mathcal{H}_t | t \in \mathrm{R}_+\}$ is right continuously nondecreasing collection of $\sigma-$algebras.By the definitions of $\mathcal{F}_t$ and $\mathcal{H}_t$,we may prove the following

Proposition 2.1 *For arbitrary $t \in R_+$, we have*

(i) $\mathcal{H}_t = \mathcal{F}_t$;

(ii) $\forall A \in \mathcal{F}_t$,*there are* $\underline{t} \in T : \underline{t} \approx t$ *and* $B \in \mathcal{B}_{\underline{t}}$,*such that* $P(A \triangle B) = 0$, *where*

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

Henceforth we assume that the filtration $\{\mathcal{F}_t | t \in \mathrm{R}_+\}$ is the standard part of the internal filtration $\{\mathcal{B}_{\underline{t}} | \underline{t} \in T\}$.Let $Q_+ = \{r_n | n \in \mathrm{N}\}$ be the set of all rational numbers in R_+.For $\underline{t} \in {}^*\mathrm{R}_+$,put

$$[\underline{t}]^T = \min\{\underline{s} \geq \underline{t} | \underline{s} \in T\}, \ \min \emptyset = \max T.$$

Lemma 2.1 *Suppose $x = \{x_t | t \in R_+\}$ is $\mathcal{F}_t-$adapted process, then there are a $\gamma \in {}^*N \setminus N$,and a process $Z : Q_+ \times \Omega \longrightarrow {}^*R$,such that for each $r \in Q_+, Z(r)$ is $\mathcal{B}_{[r+\frac{1}{\gamma}]^T}$ measurably iternal* r.v.,$^\circ Z(r) = x(r)$ a.s..

Suppose $Z : Q_+ \times \Omega \longrightarrow {}^*\mathrm{R}$ is the process obtained in Lemma 2.1.Fix arbitrarily a $m \in \mathrm{N}$,then for each $n \in \mathrm{N}, \underline{t} \in T \bigcap {}^*[0, m]$,put

$$X_n^m(\underline{t}, \cdot) = I_{{}^*[0,\frac{1}{2}]\cap T}(\underline{t}) \cdot Z(\frac{1}{2^n}, \cdot) + \sum_{k=1}^{m2^n-1} I_{{}^*(\frac{k}{2^n}, \frac{k+1}{2^n}]\cap T}(\underline{t}) \cdot Z(\frac{k+1}{2^n}, \cdot),$$

where I_A stands for the characteristic function.By Lemma 2.1,we obtain for a.a.$w \in \Omega$ and each $n \in \mathrm{N}$ that

$$^\circ X_n^m(\underline{t}, \cdot) = I_{{}^*[0,\frac{1}{2}]\cap T}(\underline{t}) \cdot x(\frac{1}{2^n}, \cdot) + \sum_{k=1}^{m2^n-1} I_{{}^*(\frac{k}{2^n}, \frac{k+1}{2^n}]\cap T}(\underline{t}) \cdot x(\frac{k+1}{2^n}, \cdot).$$

Lemma 2.2 *Let $x = \{x_t | t \in R_+\}$ be a $\mathcal{F}_t-$adapted process, $x \in RLL$.Then for each $m \in N$,there are an internal process $X_m : T \bigcap {}^*[0, m] \longrightarrow {}^*R$,and a positive infinitesimal Δ_m,such that the following hold:*

(i) For each $\underline{t} \in T \bigcap {}^*[0, m], X_m(\underline{t})$ is a $\mathcal{B}_{[\underline{t}+\Delta_m]^T}$ measurably internal r.v.;

(ii) There is a $\Omega_0 \subset \Omega : P(\Omega_0) = 1$,such that $\forall w \in \Omega_0, \underline{t} \in T \bigcap {}^*[0, m]$, $^\circ\underline{t} \leq t \Longrightarrow$ $^\circ X_m(\underline{t}, w) = x(^\circ\underline{t}, w)$; $^\circ\underline{t} > \underline{t} \Longrightarrow {}^\circ X_m(\underline{t}, w) \leq x(^\circ\underline{t}, w)$.

Theorem 2.1 *Suppose $x = \{x_t | t \in R_+\}$ is $\mathcal{F}_t-$adapted process, $x \in RLL$.Then there are a S-densely $*-$finite subset $T_0 \subset T$,an internal process $X : T_0 \times \Omega \longrightarrow {}^*R$,such that X is the SRC lifting of x.Moreover there is an internal filtration $\{\mathcal{A}_{\underline{t}} | \underline{t} \in T_0\}$,such that X is $\mathcal{A}_{\underline{t}}-$adapted,and $\{\mathcal{F}_t | t \in R_+\}$ is the standard part of $\{\mathcal{A}_{\underline{t}} | \underline{t} \in T_0\}$.*

Proof By Lemma 2.2,given arbitrarily $m \in \mathrm{N}$,there is an internal process $X_m : T \bigcap {}^*[0, m] \longrightarrow {}^*\mathrm{R}$.We may assume that (ii) in Lemma 2.2 holds for each $w \in \Omega$.Extend $\{X_m | m \in \mathrm{N}\}$ and $\{\Delta_m | m \in \mathrm{N}\}$ respectively to ${}^*\mathrm{N}$. By ω_1- saturation,there is a $\omega \in {}^*\mathrm{N} \setminus \mathrm{N}$, such that $\Delta_\omega : 0 < \Delta_\omega \approx 0$,and for each $\underline{t} \in T \bigcap {}^*[0, \omega], X_\omega^m(\underline{t})$ is a $\mathcal{B}_{[\underline{t}+\Delta_\omega]^T}$ measurably internal r.v..Put $X = X_\omega, T_0 = T \bigcap {}^*[0, \omega], \Delta = \Delta_\omega$. For each $\underline{t} \in T_0$,let $\mathcal{A}_{\underline{t}} = \mathcal{B}_{[\underline{t}+\Delta]^T}$.Now the conclusion follows from X, T_0, Δ and $\{\mathcal{A}_{\underline{t}} | \underline{t} \in T_0\}$.

After modifying the proof of Theorem 2.1 slightly,we can prove the following theorem.

Theorem 2.2 *Assume that* $x = \{x_t | t \in R_+\}$ *is a right continuously* $\mathcal{F}_t$*-adapted process.Then there are a S-densely* $*$*-finite subset* $T' \subset T$*,an internal process* $X' : T' \times \Omega \longrightarrow {}^*R$*,and an internal filtration* $\{\mathcal{A}'_{\underline{t}} | \underline{t} \in T'\}$*,such that*

(i) X' *is* $\mathcal{A}'_{\underline{t}}$*-adapted;*

(ii) For a.a.$w \in \Omega$*,and each* $\underline{t} \in ns(T')$, ${}^\circ\underline{t} \leq \underline{t} \Longrightarrow {}^\circ X'(\underline{t}, w) = x({}^\circ\underline{t}, w)$*;*

(iii) $\{\mathcal{F}_t | t \in R_+\}$ *is the standard part of* $\{\mathcal{A}'_{\underline{t}} | \underline{t} \in T'\}$*.*

3. The properties of SRC lifting

SR stands for the set of all the internal functions $F : {}^*\mathrm{R}_+ \longrightarrow \mathrm{R}$ with following conditions:

(i) For each $\underline{t} \in \mathrm{ns}({}^*\mathrm{R})$, ${}^\circ|F(\underline{t})| < +\infty$;

(ii) $\forall t \in \mathrm{R}_+$,the limits $\lim_{\underline{t} \in {}^*\mathrm{R}_+, {}^\circ\underline{t} \downarrow t} {}^\circ F(\underline{t})$ exists.

If $T' \subset T$ is a S-densely $*$-finite subset of ${}^*\mathrm{R}_+$, SR also denotes the set of all the restrictions of the elements in SR to T'.If $F : T \longrightarrow {}^*\mathrm{R}$ and $F \in SR$,we set $\mathrm{st}(F)(t) = \lim_{\underline{t} \in T, {}^\circ\underline{t} \downarrow t} {}^\circ F(\underline{t})$ for each $t \in \mathrm{R}_+$.

Theorem 3.1 *Suppose* $F : T \longrightarrow {}^*R$ *is an internal function,then* $F \in SR$ *if and only if the following hold:*

(i) If $\underline{t} \in ns(T)$, ${}^\circ|F(\underline{t})| < +\infty$*;*

(ii) For each $t \in R_+$*,there is a* $\underline{t} \in T : \underline{t} \approx t$*,such that for* $\underline{s} \in T$*,if* $\underline{s} \geq \underline{t}$ *and* $\underline{s} \approx t$*,we have* $F(\underline{s}) \approx F(\underline{t})$*.*

Proof Necessity: Let $F \in SR$,(i) is trivial.Given arbitrarily $t \in \mathrm{R}_+$,since $\lim_{\underline{t} \in T, {}^\circ\underline{t} \downarrow t} {}^\circ F(\underline{t})$ exists,we obtain that for arbitrary $\varepsilon \in \mathrm{R}_+ : \varepsilon \neq 0$,there is a $\delta \in \mathrm{R}_+$,such that

$$\forall \underline{s}_1, \underline{s}_2 \in T, 0 < {}^\circ\underline{s}_1 - t < \delta, 0 < {}^\circ\underline{s}_2 - t < \delta \Longrightarrow |{}^\circ F(\underline{s}_1) - {}^\circ F(\underline{s}_2)| < \varepsilon.$$

By saturation,there is a $H \in {}^*\mathrm{N} \setminus \mathrm{N}$,such that if put $\underline{t} = [t + \frac{1}{H}]^T$,then $\underline{t} \approx t$.Moreover $\forall \underline{s} \in T : \underline{s} \geq \underline{t}$ and $\underline{s} \approx \underline{t}$,we have $F(\underline{s}) \approx F(\underline{t})$,which implies $F \in SR$.

Sufficiency is easy by ω_1-saturation.

It is obvious that if $F \in SR$, $\mathrm{st}(F)$ is right continuous.Let $X : T \times \Omega \longrightarrow {}^*\mathrm{R}$ be an internal process,if for a.a.$w \in \Omega$, $X(\cdot, w) \in SR$,we define $\mathrm{st}(X)(t, w)$ for $t \in \mathrm{R}_+, w \in \Omega$ as follows:

$$\mathrm{st}(X)(t, w) = \begin{cases} \mathrm{st}(X(\cdot, w))(t), & X(\cdot, w) \in SR, \\ 0, & \text{otherwise}. \end{cases} \tag{1}$$

Theorem 3.2 *Assume that* $X : T \times \Omega \longrightarrow {}^*R$ *is the SRC lifting of* $x \in RLL$*, then* $st(X)(t, w) = x(t, w)$ *for all* $t \in R_+$ *and a.a.*$w \in \Omega$*.*

Proof Since X is SRC lifting of x, $\mathrm{st}(X)$ is well defined.By Definition 2.2 and (1),we obtain for a.a.$w \in \Omega$ and all $t \in \mathrm{R}_+$ the following

$$\mathrm{st}(X)(t, w) = \lim_{\underline{t} \in T, {}^\circ\underline{t} \downarrow t} {}^\circ X(\underline{t}, w) = \lim_{r \in Q_+, r \downarrow t} {}^\circ X(r, w) = \lim_{r \in Q_+, r \downarrow t} x(r, w) = x(t, w)$$

hold, which implies the theorem.

Theorem 3.2 has the following version for adapted processes.

Theorem 3.3 *Suppose the internal process $X : T \times \Omega \to {}^*R$ is $\mathcal{B}_{\underline{t}}-$adapted,and X is SRC lifting of $x \in RLL$,then x is $\mathcal{F}_t-$ adapted process.*

Proof It suffices to prove $x(t)$ is $\mathcal{F}_t$ measurable for each $t \in \mathrm{R}_+$.Since for a.a.$w \in \Omega$, $x(t,w) = \lim_{\underline{t}\in T, {}^\circ\underline{t}\downarrow t} {}^\circ X(\underline{t},w)$, so $\forall s \in \mathrm{R}_+ : s > t$,if $\underline{s} \in T$ and ${}^\circ\underline{s} = s$,${}^\circ X(\underline{s})$ is $\mathcal{H}_s$ measurable.Consequently, ${}^\circ X(\underline{t})$ is $\mathcal{H}_s$ measurable for each $\underline{t} \in T$ that satisfies $t < {}^\circ\underline{t} \le s$.Thus the limit $\lim_{\underline{t}\in T, {}^\circ\underline{t}\downarrow t} {}^\circ X(\underline{t})$ is $\mathcal{H}_s$ measurable, i.e., $x(t)$ is $\mathcal{H}_s$ measurable.Therefore $x(t)$ is $\bigcap_{s>t}\mathcal{H}_s = \mathcal{H}_t$ measurable.$\mathcal{F}_t = \mathcal{H}_t$ implies the theorem.

Recall that $U : \Omega \longrightarrow T$ is an internal stopping time with respect to the internal filtration $\{\mathcal{B}_{\underline{t}} | \underline{t} \in T\}$ if $\{U \le \underline{t}\} \in \mathcal{B}_{\underline{t}}$ for each $\underline{t} \in T$.

Theorem 3.4 *Let $T_1 \subset T$ be a S-densely internal subset, $\{\mathcal{B}_{\underline{t}} | \underline{t} \in T_1\}$ be an internal filtration with the standard part $\{\mathcal{F}_t | t \in \mathrm{R}_+\}$.Then we have*

(i) The mapping $u : \Omega \longrightarrow \overline{\mathrm{R}}_+$ is $\mathcal{F}_t-$stopping time if and only if there is an internal $\mathcal{B}_{\underline{t}}-$stopping time $U : \Omega \longrightarrow T_1$,such that ${}^\circ U = u$ a.s.;

(ii) If $X : T \times \Omega \longrightarrow {}^*R$ is SRC lifting of $x \in RLL$,then for arbitrary $\mathcal{F}_t-$stopping time u,there are an internal $\mathcal{B}_{\underline{t}}-$stopping time V,and a $P-$null set $\Omega' \subset \Omega$,such that

$$\forall w \in \Omega \setminus \Omega', {}^\circ V(w) = u(w), \text{and } \forall \underline{t} \in T, \underline{t} \approx u(w), \underline{t} \ge V(w) \Longrightarrow {}^\circ X(\underline{t}, w) = x(u(w), w).$$

Moreover if u is a constant,V may be chosen to be a constant.

Proof (i) is trivial by Theorem 4.7 in [5].It suffices to prove (ii).By Theorem 3.2,$x(t,w) = \mathrm{st}(X)(t,w)$ for a.a.$w \in \Omega$ and each $t \in \mathrm{R}_+$.Let $U : \Omega \longrightarrow T_1$ be an internal $\mathcal{B}_{\underline{t}}-$stopping time, ${}^\circ U = u$ a.s.,and Y be a lifting of $x(u)$,then

$${}^\circ Y = x(u) = \mathrm{st}(X)(u) = \mathrm{st}X({}^\circ U) \text{ a.s.}$$

Hence for a.a.$w \in \Omega$, ${}^\circ Y(w) = \lim_{\underline{t}\in T, {}^\circ\underline{t}\downarrow {}^\circ U(w)} {}^\circ X(\underline{t})$,which implies there is a sequence $\{\sigma_n | n \in \mathrm{N}\} \subset \mathrm{R}_+$,such that for each $n \in \mathrm{N}, 0 < \sigma_n < \frac{1}{n}$,moreover

$$P(\{w| \sup_{\underline{t}\in T, {}^\circ U(w) < {}^\circ\underline{t} \le {}^\circ U(w)+\sigma_n} ({}^\circ|Y(w) - X(\underline{t},w)|) \ge \frac{1}{n}\}) < \frac{1}{n}.$$

By saturation,there is a $\underline{\delta} \in T$,such that $\underline{\delta} \approx 0$ and for each $n \in \mathrm{N}$,

$$\overline{P}(\{w| \max_{\underline{t}\in T, U(w)+\underline{\delta} < \underline{t} \le U(w)+\sigma_n} (|Y(w) - X(\underline{t},w)|) \ge \frac{1}{n}\}) < \frac{1}{n}.$$

Put

$$\Omega' = \{w| \sup_{\underline{t}\in T, U(w)+\underline{\delta} < \underline{t} \approx U(w)} ({}^\circ|Y(w) - X(\underline{t},w)|) > 0\}.$$

Thus $P(\Omega') = 0$. For $w \in \Omega$,set

$$V(w) = \min\{\underline{s} \in T_1 | \underline{s} > U(w) + \underline{\delta}\}, \ \min\emptyset = \max T_1.$$

Then V satisfies the needs of the theorem.

4. Applications to optimal stopping problems

Suppose $X: T\times\Omega \longrightarrow {}^*\mathrm{R}$ is an internal process. Since T is a $*-$finite set, we may study the optimal stopping problem about X with the elementary methods. And the optimal stopping problem of the corresponding standard process may be solved.

Definition 4.1 *Let $x=\{x_t|t\in R_+\}\in RLL$, and the following conditions hold:*

(i) $E(\sup_{t\in\mathrm{R}_+}|x_t|)< +\infty$;

(ii) x is $\mathcal{F}_t-$adapted process;

(iii) There is a sequence $\{s_n|n\in N\}\subset R_+ : s_n < s_{n+1}\ (n\in N), \lim_{n\to+\infty} x(s_n) = \overline{\lim}_{t\to+\infty} x(t)$ a.s..

$L_{rs}(R_+,\mathcal{F}_t)$ denotes the set of all processes in RLL that satisfy above (i)(ii) and (iii). We let $\mathcal{L}_{SRC}(T,\mathcal{B}_{\underline{t}})$ denote the set of all internal process $X: T\times\Omega \longrightarrow {}^*R$ with the following conditions:

(i)′ There is a S-integrably internal r.v.Y, such that $\forall\underline{t}\in Y, |X(\underline{t},w)|\le Y(w)\ \overline{P}$.a.s.;

(ii)′ X is the class SRC on T, and $\mathcal{B}_{\underline{t}}-$adapted;

(iii)′ There are a S-integrably internal r.v.X_∞, and a subset $\{\underline{s}_1,\underline{s}_2,...,\underline{s}_{\eta'}\}\subset T$, $(\eta'\in {}^*N\setminus N)$, such that

$$\forall n=1,2,...,\eta'-1, \underline{s}_n<\underline{s}_{n+1}, \text{and } {}^\circ\underline{s}_n=+\infty \Longleftrightarrow n\in {}^*N\setminus N, n\le\eta',$$
$$\forall\underline{t}\in T\setminus ns(T), \forall n\in {}^*N\setminus N, n\le\eta,\ {}^\circ X(\underline{t})\le {}^\circ X_\infty,\ {}^\circ X(\underline{s}_n)={}^\circ X_\infty \text{ a.s.}$$

Lemma 4.1 *The process $x\in L_{rs}(R_+,\mathcal{F}_t)$ if and only if there are a S-densely $*-$finite subset $T_1\subset T$, an internal filtration $\{\mathcal{A}_{\underline{t}}|\underline{t}\in T_1\}$, and an internal process $X\in\mathcal{L}_{SRC}(T_1,\mathcal{A}_{\underline{t}})$, such that X is SRC lifting of x, and $\{\mathcal{F}_t|t\in R_+\}$ is the standard part of $\{\mathcal{A}_{\underline{t}}|\underline{t}\in T_1\}$.*

Remark 4.1 *If $x=\{x_t|t\in R_+\}$ and $X=\{X_{\underline{t}}|\underline{t}\in T_1\}$ satisfy the conditions of lemma 4.1, we have $\lim_{t\to+\infty} x_t = {}^\circ X_\infty$ a.s., where X_∞ satisfies (iii)′ in Definition 4.1.*

For convenience, we may from now on assume $x_\infty=\overline{\lim}_{t\to+\infty} x_t$ a.s., and the SRC lifting X of x belong to $\mathcal{L}_{SRC}(T,\mathcal{B}_{\underline{t}})$ for each $x\in L_{rs}(\mathrm{R}_+,\mathcal{F}_t)$.

Theorem 4.1 *Let $x\in L_{rs}(R_+,\mathcal{F}_t)$ and $X\in\mathcal{L}_{SRC}(T,\mathcal{B}_{\underline{t}})$ be the SRC lifting of x, V be an internal $\mathcal{B}_{\underline{t}}-$stopping time. Then ${}^\circ X(V)\le x({}^\circ V)$a.s., where $x(\infty)=x_\infty$.*

Proof we may assume that (i)(ii) in Definition 2.2,(ii)(iii) and (iii)′ in Definition 4.1 hold for each $w\in\Omega$. Therefore if ${}^\circ V(w)<+\infty$, we have

$${}^\circ V(w)\le V(w) \Longrightarrow {}^\circ X(V(w),w)=x({}^\circ V(w),w);$$
$${}^\circ V(w)> V(w) \Longrightarrow {}^\circ X(V(w),w)\le x({}^\circ V(w),w).$$

If ${}^\circ V(w)=+\infty$, ${}^\circ X(V(w),w)\le {}^\circ X_\infty(w)=x_\infty(w)$. Thus ${}^\circ X(V)\le x({}^\circ V)$ which completes the proof.

Let $\mathcal{D}$ denote the set of all internal $\mathcal{B}_{\underline{t}}-$stopping times, i.e.,

$$\mathcal{D}=\{V:\Omega\longrightarrow T|V \text{ is an internal } \mathcal{B}_{\underline{t}}-\text{stopping time}\}.$$

Obviously $\mathcal{D}$ is a $*-$finite set.$\mathcal{C}$ denotes the set of all $\mathcal{F}_t-$ stopping times,i.e.,

$$\mathcal{C} = \{u : \Omega \longrightarrow \overline{\mathrm{R}}_+ | u \text{ is a } \mathcal{F}_t - \text{stopping time}\}.$$

For $x \in L_{rs}(R_+, \mathcal{F}_t)$,set $v_x = \sup_{\tau \in \mathcal{C}}\{Ex(\tau)\}$.If there is a $\tau' \in \mathcal{C}, Ex(\tau') = v_x$,then τ' is called optimal stopping of x.

Definition 4.2 *Suppose* $X \in \mathcal{L}_{SRC}(T, \mathcal{B}_{\underline{t}})$,*and set* $V_X = \max_{U \in \mathcal{D}}\{\overline{E}X(U)\}$.*If there is a* $U' \in \mathcal{D}$,*such that* $V_X = \overline{E}X(U'), U'$ *is called S-optimal stopping of* X.

Theorem 4.2 *Assume that* $x \in L_{rs}(R_+, \mathcal{F}_t), X \in \mathcal{L}_{SRC}(T, \mathcal{B}_{\underline{t}})$ *and* X *is the SRC lifting of* x.*If* $V \in \mathcal{D}$ *is the S-optimal stopping of* X,*then* ${}^{\circ}V \in \mathcal{C}$ *is the optimal stopping of* x.

Proof It follows from Theorem 3.4 that ${}^{\circ}V \in \mathcal{C}$.We set $\tau_0 = {}^{\circ}V$ and assume that τ_0 isn't the optimal stopping of x,then there is a $\tau_1 \in \mathcal{C}$,such that $Ex(\tau_0) < Ex(\tau_1)$.Theorem 3.4 implies that there is a lifting $U_1 \in \mathcal{D}$ of τ_1,such that ${}^{\circ}X(U_1) = x(\tau_1)$ a.s..By the S-integrability of X and Theorem 4.1,we obtain

$${}^{\circ}\overline{E}X(U_1) = Ex(\tau_1) > Ex(\tau_0) \geq E({}^{\circ}V(V)) = {}^{\circ}\overline{E}X(V).$$

Thus ${}^{\circ}\overline{E}X(U_1) > {}^{\circ}\overline{E}X(V)$ implies $\overline{E}X(U_1) > \overline{E}X(V)$, which contradicts the assumption that V is the S-optimal stopping of X. Hence ${}^{\circ}V$ is the optimal stopping of x.

By Theorem 4.2,to find the optimal stopping of $x \in L_{rs}(\mathrm{R}_+, \mathcal{F}_t)$,it suffices to find the S-optimal stopping of the SRC lifting $X \in \mathcal{L}_{SRC}(T, \mathcal{B}_{\underline{t}})$ of x.We write $T = \{\underline{t}_0, \underline{t}_1, ..., \underline{t}_\gamma\}$ and assume $\underline{t}_0 < \underline{t}_1 < \cdots < \underline{t}_\gamma$,where $\gamma \in {}^*\mathrm{N} \setminus \mathrm{N}$.For $\underline{t} \in T$, set

$$\mathcal{D}_{\underline{t}} = \{V \in \mathcal{D} | V \geq \underline{t}\}. \tag{2}$$

Obviously,$\mathcal{D}_{\underline{t}}$ is a $*-$finite subset of $\mathcal{D}$.Let

$$\Gamma_{\underline{t}} = \max_{V \in \mathcal{D}_{\underline{t}}}\{\overline{E}(X(V)|\mathcal{B}_{\underline{t}})\}. \tag{3}$$

The following result is similar to one in the classical optimal stopping problem [1].

Theorem 4.3 *Suppose* $X \in \mathcal{L}_{SRC}(T, \mathcal{B}_{\underline{t}})$,*then for each* $n \in {}^*N, n \leq \gamma$,*the following*

$$\Gamma_{\underline{t}_\gamma} = X_{\underline{t}_\gamma}, \ \ \Gamma_{\underline{t}_n} = \max\{X_{\underline{t}_n}, \overline{E}(\Gamma_{\underline{t}_{n+1}} | \mathcal{B}_{\underline{t}_n})\} \ (n < \gamma) \, \overline{P} \text{ a.s.} \tag{4}$$

hold.

We'll construct a S-optimal stopping of an internal process $X \in \mathcal{L}_{SRC}(T, \mathcal{B}_{\underline{t}})$ by the hyperfinite backward induction.

Theorem 4.4 *Let* $X \in \mathcal{L}_{SRC}(T, \mathcal{B}_{\underline{t}})$,*and for each* $n \in {}^*N, n \leq \gamma$,*set*

$$U_{\underline{t}_n} = \min\{\underline{s} \in T | \underline{s} \geq \underline{t}_n, X_{\underline{s}} \geq \Gamma_{\underline{s}}\}, \ \min \emptyset = \underline{t}_\gamma.$$

Then $U_{\underline{t}_n} \in \mathcal{D}_{\underline{t}_n}$, $\overline{E}(X(U_{\underline{t}_n})|\mathcal{B}_{\underline{t}_n}) = \Gamma_{\underline{t}_n}$ $\overline{P}$.*a.s.,and* $\overline{E}\Gamma_{\underline{t}_n} = \max_{U \in \mathcal{D}_{\underline{t}_n}}\{\overline{E}X(U)\}$.

Proof If $n = \gamma$,the conclusions are trivial.Assume that the theorem holds respectively for

$n=\gamma-1,\ldots,k$. Then when $n=k-1, U_{\underline{t}_{k-1}}\in\mathcal{D}_{k-1}$.Let $V'=\max\{U_{\underline{t}_{k-1}},\underline{t}_k\}$,then $V'\in\mathcal{D}_{\underline{t}_k}$. Since $\Gamma_{\underline{t}_k}=\overline{E}(X(U_{\underline{t}_k})|\mathcal{B}_{\underline{t}_k})\ \overline{P}$.a.s.,so if setting $A_k=\{X(\underline{t}_{k-1})\geq\overline{E}(\Gamma_{\underline{t}_k}|\mathcal{B}_{\underline{t}_{k-1}})\}$,we have by the assumption of induction the following

$$\overline{E}(I_A\cdot X(U_{\underline{t}_{k-1}}))=\overline{E}(I_{A\cap A_k}\cdot X(\underline{t}_{k-1}))+\overline{E}(I_{A\cap A_k^c}\cdot X(V'))$$

hold for arbitrary $A\in\mathcal{B}_{\underline{t}_{k-1}}$.And

$$\overline{E}(I_A\cdot X(U_{\underline{t}_{k-1}}))=\overline{E}(I_{A\cap A_k}\cdot\Gamma_{\underline{t}_{k-1}})+\overline{E}(I_{A\cap A_k^c}\cdot\overline{E}(\Gamma_{\underline{t}_k}|\mathcal{B}_{\underline{t}_{k-1}}))=\overline{E}(I_A\cdot\Gamma_{\underline{t}_{k-1}}).$$

Therefore $\overline{E}(X(U_{\underline{t}_{k-1}})|\mathcal{B}_{\underline{t}_{k-1}})=\Gamma_{\underline{t}_{k-1}}\ \overline{P}$. a.s.,which is the case when $n=k-1$.It is easy to show that $E(\Gamma_{\underline{t}_n})=\max_{U\in\mathcal{D}_{\underline{t}_n}}\{EX(U)\}$.

The following result follows from Theorem 4.2 and Theorem 4.4.

Corollary 4.1 *Suppose $x\in L_{rs}(R_+,\mathcal{F}_t), X\in\mathcal{L}_{SRC}(T,\mathcal{B}_{\underline{t}})$ is SRC lifting of x,let*

$$V=\min\{\underline{t}\in T|X(\underline{t})\geq\Gamma_{\underline{t}}\}$$

then ${}^{\circ}V$ is the optimal stopping of x.

Remark 4.2 *If $x\in L_{rs}(R_+,\mathcal{F}_t), X$ is an internal process that satisfies $st(X)=x$ a.s.,but X isn't the SRC lifting of x, then Theorem 4.2 and Corollary 4.1 are false.*

Example Let $(\Omega,\mathcal{B},\overline{P})$ be a $*$–finite probability space,$\eta_0\in{}^*\mathrm{N}\setminus\mathrm{N},\eta_1=\eta_0^4,\Delta t=\frac{1}{\eta_0^2}\approx 0, T=\{k\cdot\Delta t|k\in{}^*\mathrm{N},0\leq k\leq\eta_1\}$.We define $X:T\times\Omega\longrightarrow{}^*\mathrm{R}$ and $x:\mathrm{R}_+\times\Omega\longrightarrow\mathrm{R}$ as follows:

$$X(\underline{t},w)=\begin{cases}0, & 0\leq\underline{t}<1-\eta_0\cdot\Delta t,\\ 2, & 1-\eta_0\cdot\Delta t\leq t<1,\\ \frac{1}{2}, & 1\leq\underline{t}<2,\\ 1, & 2\leq\underline{t}<\eta_0^2,\end{cases}\qquad x(t,w)=\begin{cases}0, & 0\leq\underline{t}<1,\\ \frac{1}{2}, & 1\leq\underline{t}<2,\\ 1, & 2\leq\underline{t}<+\infty.\end{cases}$$

Then $x\in L_{rs}(\mathrm{R}_+,\mathcal{F}_t)$,and $\mathrm{st}(X)(t)=x(t)$,but X is not the SRC lifting of x,where $\mathcal{F}_t\equiv\mathcal{F},\mathcal{B}_{\underline{t}}\equiv\mathcal{B}$.Hence if $\underline{t}\in{}^*[1-\eta_0\cdot\Delta t,1)\bigcap T, V\equiv\underline{t}$ is the S-optimal stopping of X,but ${}^{\circ}V\equiv 1$ is not the optimal stopping of x.The optimal stopping of x is $v\equiv t\ (t\geq 2)$.

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Loeb 空间上一类右连续过程最优停止的非标准表示

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摘　要

本文在 Loeb 空间上得到了右连续左半上连续的随机过程的 SRC 提升. 证明了一个内过程的 S- 最优停止的存在性，并得到了它的结构性表示. 最后证明了一个过程 SRC 提升的 S- 最优停止的标准部分即为对应标准过程的最优停止，在 Loeb 空间上推广了 [8] 中的结果.