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Duality for Modules of Finite Length*

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Abstract: In this paper we generalize the theory of local duality and discuss the homological dimension of modules of finite length.

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1. Preliminaries on local duality

Let (R, M) be a commutative local Noetherian ring with unique maximal ideal M and A an R-module of finite length. Let $k = R/M$ and denote by E the injective hull of k. Then the local duality (also called Matlis duality) of A is defined to be $D(A) = \mathrm{Hom}_R(A, E)$. The following results are well-known:

1) The functor $D(-) = \mathrm{Hom}_R(-, E)$ defines a duality between the category of modules of finite length and itself. This means that if the length of A is finite then so is $D(A)$, and there is a natural isomorphism between A and $D(D(A))$.

2) $\beta_i(D(A)) = \mu_i(A)$ for all $i \geq 0$, where $\beta_i(D(A))$ denotes the i-th Betti number of $D(A)$ and $\mu_i(A)$ denotes the i-th Bass number of A.

For a finitely generated module A over the local Noetherian ring R, the projective and injective dimension of A can be given by the Betti and Bass numbers:

$$\mathrm{proj\ dim}(A) = \sup\{i \mid \beta_i(A) \neq 0\}, \quad \mathrm{inj\ dim}(A) = \sup\{i \mid \mu_i(A) \neq 0\}.$$

2. General duality

In the following R denotes a commutative Noetherian ring with identity. Before proceeding we need some lemmas.

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Lemma 1 *Let $A \neq 0$ be an R-module. Then the length $L_R(A) < \infty$ if and only if* $\mathrm{Supp}(A) \subseteq \mathrm{Max}(R)$, $\mathrm{Supp}(A)$ *is a finite set, and $L_{R_M}(A_M) < \infty$ for all $M \in \mathrm{Supp}(A)$.*

Proof Suppose first that $L_R(A) < \infty$. If $P \in \mathrm{Ass}(A)$, then $R/P\ (\to A)$ has finite length, and thus P is a maximal ideal. Moreover, since $\mathrm{Ass}(A) \subseteq \mathrm{Supp}(A)$ and since the set of minimal elements of $\mathrm{Ass}(A)$ and of $\mathrm{Supp}(A)$ coincide [1, Theorem 6.5], we must have $\mathrm{Ass}(A) = \mathrm{Supp}(A)$. Thus $\mathrm{Supp}(A) \subseteq \mathrm{Max}(R)$ and $\mathrm{Supp}(A)$ is a finite set. Clearly, we have $L_{R_M}(A_M) < \infty$ for all $M \in \mathrm{Supp}(A)$.

Conversely, let $A^{(1)} \supseteq A^{(2)} \supseteq \cdots$ be a descending chain of submodules of A. Since $L_{R_M}(A_M) < \infty$ for all $M \in \mathrm{Supp}(A)$ and since $\mathrm{Supp}(A)$ is finite, there is an integer $s \geq 1$ such that $A_M^{(s)} = A_M^{(s+j)}$ for all $M \in \mathrm{Supp}(A)$ and all $j \geq 1$. But $\mathrm{Supp}(A^{(i)}) \subseteq \mathrm{Supp}(A)$ for all i, it is clear that $(A^{(s)}/A^{(s+j)})_M = 0$ for all $M \in \mathrm{Max}(R)$, and thus $A^{(s)} = A^{(s+j)}$ for all $j \geq 1$. Similarly, A satisfies the ascending chain condition, so $L_R(A) < \infty$.

Lemma 2 *Let $M, P \in \mathrm{Max}(R)$. Then*

$$E_R(R/M)_P \cong \begin{cases} E_{R_M}(R_M/MR_M), & \text{if } M = P, \\ 0, & \text{if } M \neq P, \end{cases}$$

and

$$\mathrm{Hom}_R(R/M, E_R(R/P)) \cong \begin{cases} R/M, & \text{if } M = P, \\ 0, & \text{if } M \neq P, \end{cases}$$

where $E_R(A)$ is the injective hull of the R-module A.

Proof The first assertion is a consequence of [1, Theorem 18.4(v), (vi)].

Next, let $0 \neq f \in \mathrm{Hom}_R(R/M, E_R(R/M))$. Then $\mathrm{Im} f \cap R/M \neq 0$ and so $\mathrm{Im} f \supseteq R/M$, since R/M is a simple R-module. But $\mathrm{Im} f$, as a non-zero image of a simple module, must be simple, it follows that $\mathrm{Im} f = R/M$. Thus

$$\mathrm{Hom}_R(R/M, E_R(R/M)) \cong \mathrm{Hom}_R(R/M, R/M) \cong R/M.$$

Now suppose that $M \neq P$. Then for all $Q \in \mathrm{Max}(R)$, either $(R/M)_Q = 0$ or $E_R(R/P)_Q = 0$ by the first part of the lemma. So, for all $Q \in \mathrm{Max}(R)$,

$$\mathrm{Hom}_R(R/M, E_R(R/P))_Q \cong \mathrm{Hom}_{R_Q}((R/M)_Q, E_R(R/P)_Q) = 0,$$

and thus $\mathrm{Hom}_R(R/M, E_R(R/P)) = 0$.

From now on, let $A \neq 0$ be an R-module of finite length. Thus $\mathrm{Supp}(A)$ is a finite set by Lemma 1. If $\mathrm{Supp}(A) = \{M_1, \cdots, M_n\}$, set $\overline{A} = E_R(R/M_1) \oplus \cdots \oplus E_R(R/M_n)$, and define $D(A) = \mathrm{Hom}_R(A, \overline{A})$. Furthermore, denote by D_i the local duality over R_{M_i}, namely, $D_i = \mathrm{Hom}_{R_{M_i}}(-, E_{R_{M_i}}(R_{M_i}/M_iR_{M_i}))$.

Lemma 3 $D(A)_{M_i} \cong D_i(A_{M_i})$ *for $1 \leq i \leq n$ and* $\mathrm{Supp}(D(A)) = \mathrm{Supp}(A)$.

Proof The isomorphisms $D(A)_{M_i} \cong D_i(A_{M_i})$ follow from Lemma 2. Clearly, we have $\mathrm{Supp}(D(A)) \subseteq \mathrm{Supp}(A)$. On the other hand, $D(A)_{M_i} \cong D_i(A_{M_i}) \neq 0$. Hence $\mathrm{Supp}(D(A)) = \mathrm{Supp}(A)$.

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Theorem 4 *Let assumptions and notation be as above. Then*

(1) $L_R(D(A)) < \infty$ *and* $D(D(A)) \cong A$.

(2) *For all* $j \geq 0$ *and* $1 \leq i \leq n$, $\beta_j(D(A)_{M_i}) = \mu_j(A_{M_i})$.

Proof (1) $L_R(D(A)) < \infty$ follows from Lemmas 1 and 3. We next show, using induction on $L_R(A)$, that the evaluation map $f_A : A \to \mathrm{Hom}_R(\mathrm{Hom}_R(A, \overline{A}), \overline{A})$ $(= D(D(A)))$ is an isomorphism. If $L_R(A) = 1$, we may assume that $A = R/M_1$, and we have $D(R/M_1) \cong R/M_1$ by Lemma 2. So, $D(D(A)) \cong D(A) \cong A$. It is easily checked that f_A is not zero, and thus an isomorphism.

If A has length $m > 1$, we can find a short exact sequence $0 \to A' \to A \to A'' \to 0$ with $A' \cong R/M_1$ (since $\mathrm{Ass}(A) = \mathrm{Supp}(A)$ as seen in the proof of Lemma 1). So the length of A'' is $m - 1$. Since $\overline{A}$ is injective, we get an exact sequence

$$0 \to \mathrm{Hom}_R(A'', \overline{A}) \to \mathrm{Hom}_R(A, \overline{A}) \to \mathrm{Hom}_R(A', \overline{A}) \to 0.$$

we now show that $\mathrm{Hom}_R(A', \overline{A}) \cong D(A')$ and $\mathrm{Hom}_R(A'', \overline{A}) \cong D(A'')$. We first have $\mathrm{Hom}_R(A', \overline{A}) = \oplus_{i=1}^{n} \mathrm{Hom}_R(A', E_R(R/M_i)) \cong \mathrm{Hom}_R(A', E_R(R/M_1)) \cong D(A')$ by Lemma 2. Next, it is clear that $\{M_2, \cdots, M_n\} \subseteq \mathrm{Supp}(A'') \subseteq \mathrm{Supp}(A)$. If $\mathrm{Supp}(A'') = \mathrm{Supp}(A)$, then $\mathrm{Hom}_R(A'', \overline{A}) = D(A'')$. If $\mathrm{Supp}(A'') = \{M_2, \cdots, M_n\}$, then for all $Q \in \mathrm{Max}(R)$, $\mathrm{Hom}_R(A'', E_R(R/M_1))_Q \cong \mathrm{Hom}_{R_Q}(A''_Q, E_R(R/M_1)_Q) = 0$ by Lemma 2, and thus

$$\mathrm{Hom}_R(A'', E_R(R/M_1)) = 0.$$

So $\mathrm{Hom}_R(A'', \overline{A}) \cong \mathrm{Hom}_R(A'', E_R(R/M_1)) \oplus D(A'') \cong D(A'')$. It follows that we have the exact sequence $0 \to D(A'') \to D(A) \to D(A') \to 0$.

Applying $\mathrm{Hom}_R(-, \overline{A})$ again, we get a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & A' & \to & A & \to & A'' & \to & 0 \\
 & & \downarrow f_{A'} & & \downarrow f_A & & \downarrow f_{A''} & & \\
0 & \to & D(D(A')) & \to & D(D(A)) & \to & D(D(A'')) & \to & 0
\end{array}$$

showing that f_A is an isomorphism, since both $f_{A'}$ and $f_{A''}$ are isomorphisms by induction.

(2) Use Lemma 3 and the local duality.

3. Homological dimension

Theorem 5 *Let* $A \neq 0$ *be an* R*-module of finite length. Then* $\mathrm{inj\ dim}(A) = \mathrm{proj\ dim}(D(A))$.

Proof If R is local, then

$$\mathrm{inj\ dim}(A) = \sup\{i \mid \mu_i(A) \neq 0\} = \sup\{i \mid \beta_i(D(A)) \neq 0\} = \mathrm{proj\ dim}(D(A)).$$

In the general case, by Lemma 3 and Theorem 4(2), we have

$$\begin{aligned}
\mathrm{inj\ dim}(A) &= \sup_{1 \leq i \leq n}\{\mathrm{inj\ dim}(A_{M_i})\} = \sup_{1 \leq i \leq n}\{\mathrm{proj\ dim}(D_i(A_{M_i}))\} \\
&= \sup_{1 \leq i \leq n}\{\mathrm{proj\ dim}(D(A)_{M_i})\} = \mathrm{proj\ dim}(D(A)).
\end{aligned}$$

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From Theorem 5 we get the following interesting

Corollary 6 *Let R be an Artinian ring with J its Jacobson radical. Then* inj dim (R) = proj dim $(E_R(R/J))$.

Proof Let $\mathrm{Max}(R) = \{M_1, \cdots, M_n\}$. Then

$$D(R) = \mathrm{Hom}_R(R, \overline{R}) \cong \overline{R} = \bigoplus_{i=1}^{n} E_R(R/M_i) \cong E_R(\bigoplus_{i=1}^{n} R/M_i) \cong E_R(R/J).$$

By Theorem 5 the result follows.

In [3] and [4] it is shown that proj dim (B) = inj dim (B) for all simple modules B. This can be recovered by using Theorem 5, since such modules are self-dual by Lemma 2. Finding finitely generated modules B with the property that proj dim (B) = inj dim (B) is an interesting equestion. In [5] it is shown that if (R, M) is a local ring and B is a finitely generated R-module with a simple module as its direct summand, then proj dim (B) = inj dim (B). But this is obvious from the facts that proj dim (R/M) = inj dim (R/M) = glob. dim (R), proj dim $(C \oplus D) = \max\{$proj dim (C), proj dim $(D)\}$ and inj dim $(C \oplus D) = \max\{$inj dim (C), inj dim $(D)\}$ for all R-modules C and D.

We now give a class of modules B with the property that proj dim (B) = inj dim (B).

Corollary 7 *Let $A \neq 0$ be an R-module of finite length. Then* proj dim $(A \oplus D(A))$ = inj dim $(A \oplus D(A))$.

Proof Since $D(A \oplus D(A)) \cong A \oplus D(A)$, by Theorem 5 the result follows.

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长度有限的模的对偶

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摘 要

本文推广了局部对偶理论并讨论了长度为有限的模的同调维数

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