Journal of Mathematical Research & Exposition
Vol.19, No.3, 483-489, August 1999

Fundamental Groups of Triangle Geometries *

XIA Jian-guo
(Dept. of Math., Nanjing Normal University, Nanjing 210097)

Abstract: In this paper, depending on the theory of buildings, we obtain a kind of method completely different from past ones, to compute topological fundamental groups of some triangle geometries. This method will enable us to easily compute topological fundamental groups of infinitely many finite triangle geometries.

Key words: building; covering; triangle geometry; fundamental group.

Classification: AMS(1991) 20E42, 51E24/CLC 152.1

Document code: A **Article ID:** 1000-341X(1999)03-0483-07

Ronan[5] initiated the study of triangle geometries which all arise as quotients of certain affine buildings. In [4] covering and fundamental group of a chamber system and its geometric realization were defined. In [6] Ronan proved that if a chamber system has rank 3 then its fundamental group is isomorphic to topological fundamental group of its geometric realization.

Define of the fundamental group of a topological space is very intuition, but its computation is often very complex. After some preliminary definitions in Section 1, we prove in Section 2 main Theorem 2.3, which is concerned in fundamental group of a chamber system. By using Theorem 2.3 we may easily compute topological fundamental groups of infinitely many finite triangle geometries. In Section 3 topological fundamental groups of many triangle geometries are computed.

1.Chamber systems and their geometric realizations

In [7] Tits defines a chamber system over the set I to be a set C, the elements of which are called chambers, together with a partition of C for each $i \in I$. Two chambers x and y are said to be i-adjacent if they belong to the same part of partition of C for some i.

A gallery is a finite sequence of chambers $(c_0, \cdots, c_k)$ such that c_{j-1} is adjacent to c_j for each $1 \leq j \leq k$; and we always assume $c_{j-1} \neq c_j$. The gallery is said to have type

*Received date: 1996-07-09
Foundation item: National Natural Science Foundation of China(19471070); Zhejiang Natural Science Foundation(193044)
Biography: XIA Jianguo (1963-), male, Ph.D.

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

$i_1 i_2 \cdots i_k$(a word in the free monoid on I) if c_{j-1} is i_j-adjacent to c_j. If each i_j belongs to some given subset J of I, then we call it a J-gallery.

We call C connected (J-connected) if any two chambers can be joined by a gallery (or J-gallery). The J-connected components are called residues of type J or simply J-residues.

The rank of a chamber system over I is the cardinality of I.

Given residues R and S of type J and K respectively we say S is a face of R if $R \subset S$ and $J \subset K$. If we let cotype J mean type $I - J$, then given any residue R of cotype J the following two observations are immediate:

(1) for each $K \subset J$, R has a unique face of cotype K;

(2) if S_1, S_2 are faces of R of cotype K_1 and K_2 then S_1 and S_2 have the same face of cotype $K_1 \cap K_2$.

The observations above suggest that one can regard a chamber system C of finite rank n as a CW-complex ΔC of dimension $n-1$ which is called geometric realization of C in the following way.

Associate to each residue of cotype $\{i\}$ a vertex; then associate to each residue R of cotype $\{i,j\}$ an edge(1-simplex) identifying its boundary with the vertices corresponding to the faces of R. Continue inductively, associating to each residue R of cotype $\{i_1, \cdots, i_r\}$ a simplex σ of dimension $r-1$, and identifying the faces of σ with the simplexes already associated to the faces of R. One can recover the chamber system from its geometric realization by taking the chambers to be simplexes of maximal dimension $(n-1)$ and i-adjacency to be sharing a face cotype $\{i\}$.

2. Covering and the fundamental group

By definition [4] an preserving adjacency map $\varphi{:}C \to D$ of chamber systems is called a 2-covering(or simply a covering) if it maps each rank 2 residue of C isomorphically onto a rank 2 residue of D of the same type. We say also that C is a cover of D. If C has rank 3, then the covering $\varphi{:}C \to D$ induces a topological covering $\Delta\varphi : \Delta C \to \Delta D$ of geometric realizations (see [6, 2.3]).

In any chamber system an elementary homotopy of galleries is an alteration from a gallery of the form $\gamma\omega\delta$ to $\gamma\omega'\delta$ where ω and ω' are galleries (with the same extremities) in a rank 2 residue.We say that two galleries are homotopic if one can be transformed to the other by a sequence of elementary homotopies.

If c is a chamber in a connected chamber system C, a closed gallery based at c will mean a gallery starting and ending at c. The fundamental group $\pi(C,c)$ is the set of homotopic classes $[\gamma]$ of closed galleries γ based at c, together with the binary operation $[\gamma]\cdot[\gamma'] = [\gamma\gamma']$ where $\gamma\gamma'$means γ followed by γ'; using γ^{-1} to denote the reversal of γ, one has $[\gamma]^{-1} = [\gamma^{-1}]$.

Lemma 2.1 *Let $\varphi : C \to D$ be a covering. Given a gallery γ in D starting at some chamber d and given $c \in \varphi^{-1}(d)$, there is a unique gallery $\tilde{\gamma}$ in C starting at c and with $\varphi(\tilde{\gamma}) = \gamma$.*

Proof By induction on the length of the gallery γ, we may assume it is true for galleries of shorter length than γ. Since $\varphi : C \to D$ be a covering, φ maps each rank 1 residue of C

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

isomorphically onto a rank 1 residue of D of the same type. Let $\gamma = (d_0 = d, d_1, \cdots, d_n)$ has type $i_1 i_2 \cdots i_n$, then there exists a unique chamber $c_1 \in C$ such that $\varphi(c_1) = d_1$ and c_1 is i_1-adjacent to c, and therefore the result follows by induction.

Lemma 2.2 *Let $\varphi : C \to D$ be a covering and two galleries γ_1, γ_2 in D starting at d are homotopic, then their liftings $\tilde{\gamma}_1, \tilde{\gamma}_2$ in C starting at $c \in \varphi^{-1}(d)$ must have the same end chamber.*

Proof It suffices to show the case that γ_1 and γ_2 are elementary homotopic. Let $\gamma_1 = \gamma\omega\delta$ and $\gamma_2 = \gamma\omega'\delta$, where ω and ω' are galleries in a $\{i, j\}$-residue R of D with the same extremities. By Lemma 2.1 the gallery $\gamma_i (i = 1, 2)$ has a unique lifting $\tilde{\gamma}_i$ in C starting at c. Let $\tilde{\gamma}_1 = \tilde{\gamma}\tilde{\omega}\tilde{\delta}$ where $\varphi(\tilde{\gamma}) = \gamma, \varphi(\tilde{\omega}) = \omega$ and $\varphi(\tilde{\delta}) = \delta$. Let S be a $\{i, j\}$-residue containing end chamber of $\tilde{\gamma}$ in C. Because $\varphi : C \to D$ be a covering, there exists a gallery $\tilde{\omega'}$ in S such that $\tilde{\omega'}$ and $\tilde{\omega}$ have the same extremities and $\varphi(\tilde{\omega'}) = \omega'$. By the uniqueness of lifting(Lemma 2.1) one has $\tilde{\gamma}_2 = \tilde{\gamma}\tilde{\omega'}\tilde{\delta}$. Hence $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ have the same end chamber.

Obviously if $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ are homotopic galleries in C, then $\varphi(\tilde{\gamma}_1)$, $\varphi(\tilde{\gamma}_2)$ are homotopic galleries in D.

Let G be a group, $P_1, P_2, \cdots, P_n$ be subgroups of G. Take the elements of G as chambers, and set g and h are i-adjacent if and only if $gP_i = hP_i$.

We write this chamber system as

$$(G; 1; P_1, P_2, \cdots, P_n).$$

Theorem 2.3 *Let the building $C = (G; 1; P_1, P_2, \cdots, P_n)$ be a universal cover of a chamber system $D = (H; 1; Q_1, Q_2, \cdots, Q_n)$. If a homomorphism $\varphi : G \to H$ of groups induces a covering $\varphi : C \to D$ of chamber systems, then*

$$\pi(D, 1) \cong \ker\varphi.$$

Proof Let γ be any closed gallery in D based at 1. By Lemma 2.1 the gallery γ has a unique lifting $\tilde{\gamma}$ in C starting at 1, the end chamber of $\tilde{\gamma}$ is defined to be $f(\tilde{\gamma})$.

Define $f : \pi(D, 1) \to \ker\varphi$ by

$$f([\gamma]) = f(\tilde{\gamma}).$$

By Lemma 2.2, f is well-defined. Let γ_1 and γ_2 be two closed galleries in D based at 1, $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ be their liftings in C starting at 1 respectively. Let

$$\tilde{\gamma}_2 = (1, y_1, \cdots, f(\tilde{\gamma}_2)),\ \tilde{\gamma}_2' = (f(\tilde{\gamma}_1), f(\tilde{\gamma}_1)y_1, \cdots, f(\tilde{\gamma}_1)f(\tilde{\gamma}_2)).$$

Then $\varphi(\tilde{\gamma}_2') = \varphi(\tilde{\gamma}_2)$, and thus $\tilde{\gamma}_1\tilde{\gamma}_2'$ is a lifting of $\gamma_1\gamma_2$. We have

$$f([\gamma_1] \cdot [\gamma_2]) = f([\gamma_1\gamma_2]) = f(\tilde{\gamma}_1)f(\tilde{\gamma}_2) = f([\gamma_1])f([\gamma_2]).$$

Hence f is a homomorphism.

For any $y \in \ker\varphi$, take any gallery $\tilde{\gamma}$ from 1 to y in C, then $\varphi(\tilde{\gamma})$ is a closed gallery in D based 1 and $f([\varphi(\tilde{\gamma})]) = y$. thus f is a epimorphism.

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

If $f([\gamma]) = 1$, then $\tilde{\gamma}$ is a closed gallery in C based at 1. Because C is a building, C is simply-connected(i.e., $\pi(C,1) = 1$). Let $\tilde{\gamma}_0 = (1)$, then $[\tilde{\gamma}] = [\tilde{\gamma}_0]$ and so

$$[\gamma] = [\varphi(\tilde{\gamma})] = [\varphi(\tilde{\gamma}_0)] = [(1)].$$

Therefore f is a monomorphism.

Especially, if $n = 3$ we have

$$\pi(\Delta D) \cong \ker\varphi.$$

3. Examples of fundamental groups of triangle geometries

A triangle geometry will mean a rank 3 chamber system (or its geometric realization) for which all the rank 2 residues are projective planes (generalized 3-gons).

In a triangle geometry, the vertices are the rank 2 residues, and the edges joining these vertices are the rank 1 residues. Each chamber is a 2-simplex containing three vertices and three edges. The most extreme case is a tight triangle geometry, in which there are only three vertices.

Example 3.1 Let $G =< a,b,c|a^3 = b^3 = c^3 = 1,(ab^2)^2 = b^2a,(bc^2)^2 = c^2b,(ca^2)^2 = a^2c >$, $H =< x,y|x^3 = y^3 = 1,(xy^2)^2 = y^2x >$.

By [5] Theorem (2.5), $C = (G;1;(a),(b),(c))$ is a building. H is a Frobenius group of order 21. If exists $z \in H$ satisfying $(yz^2)^2 = z^2y$ and $(zx^2)^2 = x^2z$, but $z \notin (x)$ and $z \notin (y)$. Then $D = (H;1;(x),(y),(z))$ is a tight triangle geometry. Define homomorphism $\varphi : G \to H$ which is determined by $\varphi(a) = x, \varphi(b) = y$ and $\varphi(c) = z$. Then φ induces a covering $\varphi : C \to D$ of chamber systems.

Let $xy^2 = s$, then $s^7 = 1$. By Theorem 2.3,

(1) set $yz^2 = s$, then $z = yx^2y$ i.e., $z^2yx^2y = 1$, $\pi(\Delta D)$ is a normal subgroup of G generated by c^2ba^2b;

(2) set $yz^2 = s^2$, then $z = x^2y^2$ i.e., $zyx = 1$, $\pi(\Delta D)$ is a normal subgroup of G generated by cba;

(3) set $yz^2 = s^3$, then $z = yxy^2$ i.e., $z^2yxy^2 = 1$, $\pi(\Delta D)$ is a normal subgroup of G generated by c^2bab^2;

(4) set $yz^2 = s^4$, then $z = y^2x^2$ i.e., $zxy = 1$, $\pi(\Delta D)$ is a normal subgroup of G generated by cab;

(5) set $yz^2 = s^5$, then $z = y^2xy$ i.e., $z^2y^2xy = 1$, $\pi(\Delta D)$ is a normal subgroup of G generated by c^2b^2ab;

(6) set $yz^2 = s^6$, then $z = x$, a contradiction.

In cases (1),(3) and (5), their triangle geometries are each other homeomorphic and so their topological fundamental groups are each other isomorphic.

By [2] Theorem 4, G is isomorphic to the subgroup $U =< \alpha,\beta,\gamma >$ of $\mathrm{GL}(3,R)$, where $R = GF(2)[d,u^{-1}]$ and $u = 1 + d + d^2$, and

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \gamma = u^{-1}\begin{pmatrix} u & u & ud \\ 1+d & d+d^2 & d \\ d & 1+d & 1 \end{pmatrix}$$

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

The isomorphism g from G to U is determined by

$$g(a) = \alpha, g(b) = \beta^2, g(c) = \gamma^2.$$

Therefore we have

In cases (1),(3),(5), $\pi(\Delta D)$ is isomorphic to the normal subgroup of U generated by

$$u^{-1}\begin{pmatrix} ud & 0 & u \\ u & 1+d^2 & d+d^2 \\ 0 & 1 & 1+d \end{pmatrix}.$$

In cases (2), $\pi(\Delta D)$ is isomorphic to the normal subgroup of U generated by

$$u^{-1}\begin{pmatrix} u & ud & ud \\ d^2 & 1+d^2 & d^2 \\ d & d+d^2 & 1+d^2 \end{pmatrix}.$$

In cases (4), $\pi(\Delta D)$ is isomorphic to the normal subgroup of U generated by

$$u^{-1}\begin{pmatrix} ud & 0 & u(1+d) \\ 1+d^2 & 1 & 0 \\ d+d^2 & 1+d & u \end{pmatrix}.$$

Furthermore we can reduce modulo ideals of R and produce infinitely many finite triangle geometries. In fact any ideal L of R such that R/L is a nonzero finite ring induces-via reducing the coefficients of α, β, γ modulo L - a homomorphism φ_L from U onto a subgroup of SL$(3, R/L)$, then $D_L = (\varphi_L(U); 1; \varphi_L(<\alpha>), \varphi_L(<\beta>), \varphi_L(<\gamma>))$ is a triangle geometry and $\pi(\Delta D_L)$ is isomorphic to kerφ_L.

Example 3.2 $G = \langle a, b, c | a^9 = b^9 = c^9 = 1, (ab^8)^2 = b^8 a, (bc^8)^2 = c^8 b, (ca^8)^2 = a^8 c\rangle$, $H = \langle x, y | x^9 = y^9 = 1, (xy^8)^2 = y^8 x\rangle$.

By [8], $C = (G; 1; (a), (b), (c))$ is also a building. H is a Frobenius group of order $9 \cdot 73$. Similar to Example 3.1, if exists $z \in H$ satisfying $(yz^8)^2 = z^8 y$ and $(zx^8)^2 = x^8 z$, but $z \notin (x)$ and $z \notin (y)$, then $D = (H; 1; (x), (y), (z))$ is a tight triangle geometry. Define homomorphism $\varphi : G \to H$ which is determined by $\varphi(a) = x, \varphi(b) = y$ and $\varphi(c) = z$. Then φ induces a covering $\varphi : C \to D$ of chamber systems.

Let $xy^8 = s$, then $s^{73} = 1$. Set $yz^8 = s^i$ where $i \in \{1, 2, \cdots, 71\}$, we can express z by x and y, and thus obtain the fundamental group $\pi(\Delta D)$ by Theorem 2.3. For examples:

(1) set $yz^8 = s$, then $z = yx^8 y$ i.e., $z^8 yx^8 y = 1$, $\pi(\Delta D)$ is a normal subgroup of G generated by $c^8 ba^8 b$;

(2) set $yz^8 = s^2$, then $z = x^8 y^2$ i.e., $zy^7 x = 1$, $\pi(\Delta D)$ is a normal subgroup of G generated by $cb^7 a$;

(3) set $yz^8 = s^3$, then $z = yx^7 y^2$ i.e., $z^8 yx^7 y^2 = 1$, $\pi(\Delta D)$ is a normal subgroup of G generated by $c^8 ba^7 b^2$;

(4) set $yz^8 = s^4$, then $z = y^8 x^8 y^3$ i.e., $zy^6 xy = 1$, $\pi(\Delta D)$ is a normal subgroup of G generated by $cb^6 ab$.

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

Let $k = GF(8) = GF(2)[e]$ where $e^3 + e + 1 = 0$, and let R be the ring $k[d, u^{-1}]$ where d is a indeterminate and $u = 1 + d + d^2$. Define $\lambda \in \mathrm{aut}R$ which is determined by $\lambda(e) = e^2$ and $\lambda(d) = d$. Then $\lambda^3 = \mathrm{id}_R$. Define $\alpha, \beta, \gamma \in \mathrm{SL}(3, R) \propto (\lambda)$ and $\theta \in \mathrm{GL}(3, R)$ by

$$\alpha = \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & e \end{pmatrix}, \ \theta = \begin{pmatrix} 0 & 0 & u \\ e^2 + d & 1 + ed & e^2 \\ e^2 & e^2 + e + d & 1 + ed \end{pmatrix}, \ \beta = \alpha^\theta, \ \gamma = \beta^\theta.$$

Then $\alpha^9 = \beta^9 = \gamma^9 = 1, (\alpha\beta^8)^2 = \beta^8\alpha, (\beta\gamma^8)^2 = \gamma^8\beta, (\gamma\alpha^8)^2 = \alpha^8\gamma$ and θ^3 is a scalar matrix. Let $U =< \alpha, \beta, \gamma >$, then $U \leq \mathrm{SL}(3, R) \propto (\lambda)$. By [8] G is isomorphic to the U. Any ideal L of R such that R/L is a nonzero finite ring induces-via reducing the coefficients of α, β, γ modulo L - a homomorphism φ_L from U onto a subgroup of $\mathrm{SL}(3, R/L) \propto (\lambda)$, then $D_L = (\varphi_L(U); 1; \varphi_L(\langle\alpha\rangle), \varphi_L(\langle\beta\rangle), \varphi_L(\langle\gamma\rangle))$ is a triangle geometry and $\pi(\Delta D_L)$ is isomorphic to $\ker\varphi_L$.

Example 3.3 $G = \langle a, b, c | a^3 = b^3 = c^3 = 1, (ab)^2 = ba, (bc)^2 = cb, (ca)^2 = ac\rangle$. $H = \langle x, y, z | x^3 = y^3 = z^3 = 1, (xy)^2 = yx, (yz)^2 = zy, (zx)^2 = xz, (xy^2z)^2 = 1\rangle$.

By [5] Theorem (2.5), $C = (G; 1; (a), (b), (c))$ is a building. H is isomorphic to $L_3(2)$(see [1] Lemma 2) and $D = (H; 1; (x), (y), (z))$ is a triangle geometry. Define homomorphism $\varphi : G \to H$ which is determined by $\varphi(a) = x, \varphi(b) = y$ and $\varphi(c) = z$. Then φ induces a covering $\varphi : C \to D$ of chamber systems.

By [1] Theorem 8, G is.isomorphic to the subgroup $U =< \alpha, \beta, \gamma >$ of $\mathrm{GL}(3, R)$, where

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ d-1 & -1 & -1 \end{pmatrix}, \ \beta = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & -d \\ 0 & 0 & 1 \end{pmatrix},$$

$$\gamma = \begin{pmatrix} \varphi & \varphi & \varphi \\ -1 & -d & 2^{-1}(2-d) \\ 0 & 1 & 0 \\ -2^{-1}(d+1) & 0 & 0 \end{pmatrix}$$

and d is a unit in Z_2 satifying $d^2 + d + 2 = 0$, R is the subrings of Q_2 generated by $2^{-1}, d$.

Therefore we have $\pi(\Delta D)$ is isomorphic to the normal subgroup of U generated by

$$2^{-1} \begin{pmatrix} -1 & -d-2 & 2^{-1}(-3d+2) \\ 2(1-d) & 3-d & 2 \\ 1 & 1 & 2^{-1}(3d+2) \end{pmatrix}.$$

If p is an odd prime such that -7 is not a square in $GF(p)$–this happens exactly for $p \equiv 3, 5, 6 \pmod 7$–then there is a ring homomorphism from R onto $\mathrm{GF}(p^2)$ such that the automorphism becomes the involutory automorphism of $\mathrm{GF}(p^2)$. Hence it induces a group homorphism φ_p from U into $\mathrm{SU}_3(p)$. If p is an odd prime such that -7 is a square then we have a ring homomorphism from R onto $\mathrm{GF}(p)$ inducing a group homomorphism φ_p from U into $\mathrm{SL}_3(p)$. $D_p = (\varphi_p(U); 1; \varphi_p(\langle\alpha\rangle), \varphi_p(\langle\beta\rangle), \varphi_p(\langle\gamma\rangle))$ is a triangle geometry and $\pi(\Delta D_p)$ is isomorphic to $\ker\varphi_p$, where $\varphi_p(U)$ is isomorphic to $\mathrm{SU}_3(p)$ or $\mathrm{SL}_3(p)$(see

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.

[1]).

Acknowledgment The author is greatly indebted to Prof. Gan Danyan and Prof. Li Huiling for their careful reading of this paper and detailed suggestions for improvement.

References

[1] Köhler P, Meixner Th and Wester M. *The 2-adic affine building of type $\tilde{A}_2$ and its finite projections* [J]. JCT(A), 1985, **38**: 203–209.

[2] Köhler P, Meixner Th and Wester M. *The affine building of type $\tilde{A}_2$ over a local field of characteristic two* [J]. Arch, Math., 1984, **42**: 400–407.

[3] Köhler P, Meixner Th and Wester M. *Triangle groups* [J]. Comm. in Alg., 1984, **12**: 1595–1626.

[4] Ronan M A. *Lectures on Buildings* [C]. Prespectives in Mathematics 7, Academic Press, 1989.

[5] Ronan M A. *Triangle geometries* [J]. JCT(A), 1984, **37**: 294~319

[6] Ronan M A. *Coverings and automorphisms of chamber systems* [J]. Europ.J.Comb., 1980, **1**: 259–269.

[7] Tits J. *A local approach to buildings, in "The Geometric Vein(Coxter Festschrift)"* [C]. Springer-Verlag, New York/Berlin, 1981, 519–547.

[8] Xia Jianguo. *Triangle Buildings and Triangle Groups over* $GF(8)$ [J]. Algebra Colloquium, 1999, 4.

三角几何的基本群

夏 建 国

(南京师范大学数学系, 210097)

摘 要: 利用 Building 理论获得一种计算某些三角几何的基本群的新的方法. 这种方法能够容易地计算出无限多有限三角几何的基拓扑基本群.

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.