Journal of Mathematical Research & Exposition
Vol.19, No.4, 633-640, November, 1999

# Some Classes of Meta Commutative Po-Semigroups *

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**Abstract**: Results of weakly commutative poe-semigroups are extended to pseudo-commutative po-semigroups. We prove that pseudo-commutative semigroups can be decomposed into semilattices of Archimedean po-semigroups and such decomposition is not unique.

**Key words**: po-semigroups; filters; semilattice congruence; poe-semigroups; Archimedean semigroups; weakly commutativity; pseudo-commutativity; cyclic-commutativity.

**Classification**: AMS(1991) 20M17/CLC O152.7

**Document code**: A    **Article ID**: 1000-341X(1999)04-0633-08

## 1. Introduction

A semigroup $S$ endowed with a partial ordering "$\leq$" is called a po-semigroup if the multiplication of $S$ is compatible with "$\leq$", that is, $a \leq b$ implies that $xa \leq xb$ and $ax \leq bx$ for all $x \in S$. Po-semigroups with a greatest element $e$ are called poe-semigroups. Poe-semigroups were studied by Kehayopulu in [3],[5] and [6]. Recently, some results of Kehayopulu have been extented to weakly commutative po-semigroups by Jing and Chen in [2]. It was proved in [3] that all weakly commutative poe-semigroups are semilattices of Archimedean semigroups. In this paper, we amplify and strength the above result on pseudo-commutative po-semigroups and prove that all such po-semigroups are semilattices of Archimedean po-semigroups. We also demonstrate that the semilattice decomposition of a po-semigroup into Archimedean po-semigroups is not unique. This answers a problem posed by Kehayopulu in [3]. For terminology and definitions not given in this paper, the reader is referred to M.Petrich[9] and N.Kehayopulu[3].

## 2. Notations and Definitions

In this section, we give some basic definitions and notations which will be used in this paper.

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**Definition 2.1** *A subsemigroup $T$ of a po-semigroup $S$ is called Archimedean if for all $a, b \in T$ there exists $n \in N = \{1, 2, \cdots\}$ such that $a^n \leq xby$ for some $x, y \in T$.*

Examples of Archimedean po-semigroups can be found in [4].

**Definition 2.2** *A congruence $\sigma$ on a po-semigroup $S$ is called a semilattice congruence on $S$ if and only if $(a^2, a) \in \sigma$ and $(ab, ba) \in \sigma$ for all $a, b \in S$.*

**Definition 2.3** *A subsemigroup $F$ of a po-semigroup $S$ is called an order filter of $S$ if the following conditions are statisfied:*

*(i) $a, b \in S$ and $ab \in F \Rightarrow a \in F$ and $b \in F$;*

*(ii) $a \in F$ and $c \in S, c \geq a \Rightarrow c \in F$.*

**Notation 2.4** *Denote the smallest filter containing an element $x$ of a po-semigroup $S$ by $N(x)$. Call $N(x)$ the principal order filter generated by $x$. A principal order filter $N(x)$ is called strictly simple if $N(x) = \{t \in S | x \leq t\}$.*

In the case of semilattices, all principal order filters are strictly simple. However, in po-semigroups, there are principal order filters which are not necessary strictly simple [5]. It was shown by Kehayopulu in [5] that some properties of poe-semigroups can be discribed via the structure of its principal order filters.

**Notation 2.5** *Let $S$ be a po-semigroup. Let $\mathcal{N} = \{(x, y) | N(x) = N(y)\}$. Then it is well known that $\mathcal{N}$ is the semilattice congruences on $S$*[9]*. This result is rather useful in studying semilattice decompositions of semigroups.*

**Notation 2.6** *Let $SC(S)$ be the collection of all semilattice congruences on a po-semigroup $S$. For any $\sigma \in SC(S)$, denote the congruence class of $x \in S$ by $(x)_\sigma$. Define "$\prec$" by $(x)_\sigma \prec (y)_\sigma \Leftrightarrow (x)_\sigma = (xy)_\sigma$ on the quotient semigroup $S/\sigma = \{(x)_\sigma | x \in S\}$.*

It is well known that $(x)_\sigma$ is a subsemigroup of $S$ and $[(S/\sigma, \cdot, \prec)]$ is again a po-semigroup.

## 3. Weakly commutativity and pseudo-commutativity

**Definition 3.1**[3] *A poe-semigroup $S$ is called weakly commutative if for all $x, y \in S$, there exists an integer $n \in N = \{1, 2, \cdots\}$ such that $(xy)^n \leq yex$, where $e$ is the greatest element with respect to the partial ordering "$\leq$".*

The concept of weakly commutative poe-semigroups was extended to weakly commutative po-semigroups by Jing and Chen in [2]. They give the following definition.

**Definition 3.2**[2] *A po-semigroup $S$ is called meta commutative if for every $x, y \in S$ there exists a natural number $n$ such that $(xy)^n \in (ySx]$, equivalently, $(xy)^n \leq yax$ for some $a \in S$.*

**Note** Jing and Chen[2] still called the above po-semigroups weakly commutative po-semigroups. In order to make it distinct from the weakly commutativity defined by Kehayopulu[3] for poe-semigroup, we call the above po-semigroups meta comutative. Clearly, for poe-semigroups, meta commutativity means weakly commutativity as $e$ is the greatest element with respect to the partial ordering "$\leq$".

Inspired by the above definitions, we make the following definition for po-semigroups.

**Definition 3.3** *A po-semigroup $S$ is called pseudo-commutative if for any $x, y \in S$, there exists $n \in N = \{1, 2, \cdots\}$ such that $(xy)^n \leq yx^n$.*

Clearly, a pseudo-commutative po-semigroup, is a special case of a meta commutative po-semigroup when $n \geq 2$.

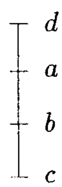
**Definition 3.4** *A po-semigroup $S$ is called weakly pseudo-commutative if for any $x, y \in S$, there exists some $n \geq 1$ such that $(xy)^n \leq yx^\lambda$ for a given integer $\lambda \geq 1$, where $\lambda$ is not necessarily equal to $n$. If $\lambda = 1$ and $x \neq y$, then we call this sort of weakly pesudo-commutative po-semigroups 2-cyclic-commutative*[11].

Clearly, 2-cyclic-commutativety does not imply 3-cyclic-commutativity. In general, $k$-cyclic-commutativity does not imply $(k+1)$-cyclic-commutativity and vice versa. Also, $k$-cyclic-commutative po-semigroup is a special case of meta commutative po-semigroup when $k \geq 3$ (We leave the details of the checking to the reader).

The following example illustrates that the class of pseudo-commutative po-semigroups is a proper subclass of the class of meta commutative po-semigroups.

**Example 3.5** Let $S$ be a po-semigroup with Hasse diagram and Cayley table as follows:

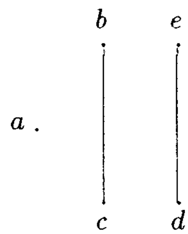
| $\cdot$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $b$ | $b$ | $a$ |
| $c$ | $a$ | $c$ | $c$ | $a$ |
| $d$ | $a$ | $a$ | $a$ | $d$ |




Then we can check that $S$ is meta commutative, however, $S$ is not pesudo-commutative, for instance, $(cb)^n \not\leq bc^n$ for all integer $n \geq 1$.

**Example 3.6** The following po-semigroup is a pseudo-commutative po-semigroup but is not 2-cyclic-commutative, for instance $(de)^2 \leq ed^2$, but $(de)^n \not\leq ed$ for all $n \in N = \{1, 2, \cdots\}$.

| $\cdot$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $a$ | $b$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $b$ | $b$ | $b$ | $b$ |
| $c$ | $a$ | $b$ | $b$ | $b$ | $b$ |
| $d$ | $a$ | $b$ | $b$ | $b$ | $b$ |
| $e$ | $a$ | $b$ | $c$ | $d$ | $e$ |




We now consider the commutativity of poe-semigroups. We have the following results.

**Proposition 3.7** *Any pseudo-commutative poe-semigroup is weakly commutative.*

**Proof** Let $S$ be a pseudo-commutative poe-semigroup. Since $e$ is the greatest element of $S$, we have $x \leq e$ and $y \leq e$ for all $x, y \in S$. This implies that $xy \leq ey$ and $xy \leq$

$xe$. Consequently, there exists $k \geq 1$ such that $(xy)^k \leq (ey)^k \leq ye^k$, by the pseudo-commutativity of $S$. Similarly, $(xy)^\ell \leq (xe)^\ell \leq ex^\ell$. Thus, $(xy)^{k+\ell} \leq ye^k ex^\ell = y(e^k e)x \leq yex$, since $e$ is the greatest element of $S$. This shows that $S$ is weakly commutative.

**Remark 3.8** From the above proof, it can be easily seen that $S$ is meta commutative as well. However, the converse is not true unless $e$ is also the multiplicative idenitity. As under such circumstance, it is not difficult to show that $S$ is weakly commutative if and only if $S$ is 2-cyclic-commutative. Thus, all the conditions of commutatives are equivalent if the greatest element $e$ of the poe-semigroup $S$ is the multiplicative identity of $S$. Since the proof is rather routine, we omit the proof.

**Proposition 3.9** *Let $S$ be a poe-semigroup. If every principal order filter of $S$ is strictly simple, then $S$ is weakly-commutative, meta commutative, pseudo-commutative, weakly pseudo-commutative and 2-cyclic-commutative.*

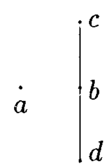
**Proof** Let $x, y \in S$. Obviously $xy \in N(xy)$, the principal order filter generated by $xy$. Then $x \in N(xy)$ and $y \in N(xy)$, by the definition of filter. Since $N(xy)$ is a filter and $e$ is the greatest element of $S, e \in N(xy)$ as well. Consequently $yex \in N(xy)$. Since $N(xy)$ is strictly simple, $xy \leq yex$. Also, since $yx \in N(xy)$, so $xy \leq yx$ by the strictly simplicity of $N(xy)$. Thus for any $k \geq 2$, we have $(xy)^k \leq (yx)^k = (yx)(yx)^{k-2}(yx) = y(x(yx)^{k-2}y)x$. Let $s = x(yx)^{k-2}y \in S$. Then $(xy)^k \leq ysx \leq yex$. This shows that $S$ is meta commutative and weakly commutative. Furthermore, since every principal ordered filter is strictly simple, we have $yex \in N((xy)^k)$. This leads to $y \in N((xy)^k)$ and $x^\lambda \in N((xy)^k)$ for all $\lambda \geq 1$, by the definition of filter. Hence $yx^\lambda \in N((xy)^k)$. Consequently, $(xy)^k \leq yx^\lambda$ as $N((xy)^k)$ is strictly simple. This proves that $S$ is pseudo-commutative, weakly pseudo-commutative and in particular, 2-cyclic commutative.

**Remark 3.10** (i) The converse of Proposition 3.9 does not hold. There are pseudo-commutative po-semigroups in which their principal ordered filters need not be strictly simple. Such examples can be found in [6].

(ii) Statments of Proposition 3.9 still hold for general po-semigroups, except the case of weakly commutativity. The following is a non-trivial po-semigroup in which all its principal order filters are strictly simple, yet it is even not a semilattice.

**Example 3.11** Let $S = \{a, b, c, d\}$ be the set with the following Cayley table and Hasse diagram:

| $\cdot$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $c$ | $d$ | $c$ | $d$ |
| $b$ | $b$ | $d$ | $b$ | $d$ |
| $c$ | $c$ | $d$ | $c$ | $d$ |
| $d$ | $d$ | $d$ | $d$ | $d$ |




We leave the reader to verifty that $S$ is a po-semigroup. It is now easy to see in this example that all principal filters of $S$ are strictly simple. In this example, we observe that $N(a) = \{a, c\}$ and $\{t \in S | t \geq a\} = \{a\}$. Thus, $N(a) \neq \{t \in S | t \geq a\}$. But $S$ is clearly

3-cyclic-commutative.

We now give a condition for $k$-cyclic-commutativity to be $(k+1)$-cyclic-commutativity.

**Proposition 3.12** *Let $S$ be a $k$-cyclic commutative po-semigroup. If $S$ is weak pseudo-commutative then $S$ is $(k+1)$-cyclic-commutative.*

**Proof** Since $S$ is $k$-cyclic commutative, we have $(a_1a_2\cdots a_k)^\lambda \leq a_k\cdots a_2a_1$ for some integer $\lambda \geq 1$, where $a_1, a_2, \cdots, a_k$ are distinict elements of $S$. Consider $((a_1a_2\cdots a_k)a_{k+1})$. By the weakly pseudo-commutativity of $S$, for the given integer $\lambda \geq 1$, there exists an integer $n \geq 1$ such that $((a_1a_2\cdots a_k)a_{k+1})^n \leq a_{k+1}(a_1\cdots a_k)^\lambda$. Then, $(a_1a_2\cdots a_ka_{k+1})^n \leq a_{k+1}(a_ka_{k-1}\cdots a_2a_1)$ by the $k$-cyclic commutativity. This shows that $S$ is $(k+1)$-cyclic-commutative.

## 4. Filters and Archimedean semigroups

In this section, we study the semilattice decomposition of po-semigroups. We start with a characterization for Archimedean po-semigroups.

**Lemma 4.1** *Let $\sigma$ be a semilattice congruence on a po-semigroup $S$. Then the $\sigma$-class $(a)_\sigma$ of $a \in S$ is an Archimedean semigroup if and only if $y \in (a)_\sigma \Rightarrow$ there exists $n \in N = \{1, 2, \cdots\}$ such that $y^n \leq uav$, for some $u, v \in (a)_\sigma$.*

**Proof** We only prove the sufficiency part as the necessity part is obvious. Let $a, b \in (x)_\sigma$. Then $(a)_\sigma = (b)_\sigma = (x)_\sigma$. This implies that $a \in (x)_\sigma = (b)_\sigma$. By the given condition, we immediately have $a^n \leq ubv$ for some $n \in N = \{1, 2, \cdots\}$ and $u, v \in (x)_\sigma$. Thus, the semigroup $(x)_\sigma$ is Archimedean.

The following theorem discribes the principal ordered filters of pseudo-commutative po-semigroups.

**Theorem 4.2** *Let $S$ be a pseudo-commutative a po-semigroup. Define $T(x) = \{a \in S | x^k \leq uav, \exists k \in N = \{1, 2, \cdots\}, \exists u, v \in N(x)\}$. Then $T(x) = N(x)$.*

**Proof** Clearly, $T(x) = \{t \in S | x^k \leq utv, \exists k \in N = \{1, 2, \cdots\}, \exists u, v \in N(x)\}$ is a subset of $S$. To prove that $T(x) = N(x)$, we only need to prove that $T$ is also a principal filter containing $x$. Clearly, $x \in T$ and so $T \neq \emptyset$.

(i) $T(x)$ is a subsemigroup of $S$. In fact, let $a, b \in T(x)$, then, by the definition of $T(x)$, we have $x^n \leq u_1av_1, x^m \leq u_2bv_2$ for some $m, n \in N = \{1, 2, \cdots\}$ and some $u_1, v_1, u_2, v_2 \in N(x)$. Since $S$ is pseudo-commutative, we have $(u_1av_1)^k = ((u_1a)v_1)^k \leq v_1(u_1a)^k, \exists k \in N = \{1, 2, \cdots\}$; $(u_2bv_2)^\ell = (u_2(bv_2))^\ell \leq (bv_2)u_2^\ell, \exists \ell \in N = \{1, 2, \cdots\}$. Hence,

$$x^{nk} \leq (u_1av_1)^k \leq v_1(u_1a)^k = v_1(u_1a)^{k-1}u_1a = v_1u_1'a, \text{ where } u_1' = (u_1a)^{k-1}u_1. \tag{1}$$

Similarly,

$$x^{m\ell} \leq (u_2bv_2)^\ell \leq (bv_2)u_2^\ell = b(v_2u_2^\ell) = bv_2' \text{ where } v_2' = v_2u_2^\ell. \tag{2}$$

The inequality (1) implies that $v_1u_1'a \in N(x^{nk}) \subset N(x)$. Hence, $v_1u_1'$ is in $N(x)$. Similarly, we have $bv_2' \in N(x)$ and so $v_2' \in N(x)$. Now, from (1) and (2), we have $x^{nk+m\ell} \leq$

$(v_1u_1')(ab)(v_2')$ with $nk+m\ell \in N = \{1,2,\cdots\}, v_1u_1' \in N(x)$ and $v_2' \in N(x)$. This implies that $ab \in T$.

(ii) Let $a \in T$ and $a \le b$ for some $b \in S$. Then, $x^n \le uav$ for some $n \in N = \{1,2,\cdots\}$, and some $u,v \in N(x)$. As $a \le b, uav \le ubv$ and thereby $x^n \le ubv$. This shows that $b \in T(x)$.

(iii) Let $ab \in T(x)$. Then $x^k \le u(ab)v$ for some $k \in N = \{1,2,\cdots\}$ and some $u,v \in N(x)$. As $utv \in N(x^k) \subset N(x)$ for all $t \in T(x)$ and $N(x)$ is a principal filter of $x$, we have $t \in N(x)$ for all $t \in T(x)$. Thus $T(x) \subseteq N(x)$ and hence $ab \in N(x)$. Consequently $a \in N(x)$ and $b \in N(x)$. This implies that $ua \in N(x)$ and $bv \in N(x)$. By $x^k \le u(ab)v = (u)a(bv) = (ua)b(v)$, we have $a \in T(x)$ and $b \in T(x)$.

Thus, by (i), (ii) and (iii), the defined subset $T(x)$ of $S$ is indeed a principal filter of $S$ containing $x \in S$. It is easy to see that $N(x) \subseteq T(x)$. Thus $N(x) = T(x)$.

**Note** (i) The converse of Theorem 4.2 does not hold. This part is different from the corresponding results in [2] and [3].

(ii) It is noticed that if $S$ is a meta commutative po-semigroup, then by using the similar arguments, we can easily prove that $N(x) = \{y \in S | x^k \le ysy$ for some $k \ge 1$ and $s \in S\}$. If the po-semigroup $S$ has a greatest element $e$, then we can also show that $N(x)$ is of the form $\{y \in S | x^k \le yey, \exists n \in N = \{1,2,\cdots\}\}$. Hence, we have the following results:

**Corollary 4.3** *(i)*[2] *A po-semigroup $S$ is meta commutative if and only if $N(x) = \{y \in S | x^k \le ysy, \exists n \in N = \{1,2,\cdots\}, s \in S\}$.*

*(ii)*[3] *A poe-semigroup $S$ is weakly commutative if and only if $N(x) = \{y \in S | x^k \le yey, \exists n \in N = \{1,2,\cdots\}\}$, for all $x \in S$.*

The following lemma is crucial in proving the main theorem:

**Lemma 4.4**[5] *Let $\mathcal{N}$ be a semilattice congruence on a po-semigroup $S$. Then $a,b \in S, a \le b \Rightarrow (a,ab) \in \mathcal{N}$.*

**Theorem 4.5** (Main theorem) *All pseudo-commutative po-semigroups can be decomposed into semilattices of Archimedean po-semigroups.*

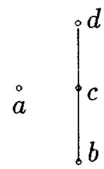
**Proof** Let $S$ be a po-semigroup and let $\mathcal{N} = \{(x,y) | N(x) = N(y)$, for $x,y \in S\}$. As $\mathcal{N} \in SC(S), (x)_{\mathcal{N}}$ is a semigroup for every $x \in S$. We now prove that each $(x)_{\mathcal{N}}$ is an Archimedean po-semigroup. For this purpose, let $b \in (x)_{\mathcal{N}}$. Then $(x)_{\mathcal{N}} = (b)_{\mathcal{N}}$, that is $N(x) = N(b)$ and so $b \in N(x)$. By Theorem 4.2, $N(x) = T(x)$, so $x^m \le ubu'$ for some integer $m \ge 1$ and some $u,u' \in N(x)$. As $x \in N(x) = N(b), x^k \in N(b)$ for all $k \in N = \{1,2,\cdots\}$, in particular, we have $x^{m+3} \in N(b)$. By applying Theorem 4.2 again, $N(b) = T(b)$ and hence $b^n \le vx^{m+3}v'$ for some $n \in N = \{1,2,\cdots\}$ and some $v,v' \in N(b) = N(x)$. Then, by the properties of semilattice congruence and Lemma 4.4, we can deduce that $(x)_{\mathcal{N}} = (b)_{\mathcal{N}} = (b^n)_{\mathcal{N}} = (b^nvx^{m+3}v')_{\mathcal{N}} = (bvxv')_{\mathcal{N}} = (v'bvx)_{\mathcal{N}}$ and $(x)_{\mathcal{N}} = (x^m)_{\mathcal{N}} = (x^mubu')_{\mathcal{N}} = (xubu')_{\mathcal{N}}$. Hence, $v'bvx \in (x)_{\mathcal{N}}$ and $(xubu') \in (x)_{\mathcal{N}}$. Since $S$ is pseudo-commutative, we have $(bvx^{m+3}v')^k \le v'(bvx^{m+3})^k$ for some $k \in N = \{1,2,\cdots\}$. As $b^n \le vx^{m+3}v'$, we have $b^{n+1} \le bvx^{m+3}v'$. This implies that $b^{(n+1)k} \le (bvx^{m+3}v')^k \le v'(bvx^{m+3})^k$ with $bv \in N(b) = N(x)$. Again, from $x^m \le ubu'$, we deduce that $x^{m+3} \leq x^3ubu'$. Consequently, $b^{(n+1)k} \leq v'(bvx^{m+3})^k \leq v'(bvx^3ubu')^k = v'[(bv)x^3ubu']^k \leq v'x^3ubu'(bv)^k$. Let $v^* = ubu'(bv)^k$. Then we have $b^{(n+1)k} \leq v'x^3v^* = (v'x)x(xv^*)$, where $(n+1)k = \lambda \in N = \{1, 2, \cdots\}$. Since $(v'x)x(xv^*) \in N(b^\lambda) \subset N(b)$, we infer that both $v'x$ and $xv^*$ are in $N(b) = N(x)$. So, from $b \in (b)_{\mathcal{N}} = (x)_{\mathcal{N}}$, we prove that there exists $\lambda \geq 1$ such that, $b^\lambda \leq (v'x)x(xv^*)$, with $v'x$ and $xv^* \in (x)_{\mathcal{N}}$. Hence, $(x)_{\mathcal{N}}$ is an Archimedean po-semigroup. The proof is completed.

**Remark 4.6** The semilattice congruence $\mathcal{N}$ on the po-semigroup $S$ in Theorem 4.5 is the greatest congruence in $S$. For if $\sigma \in \mathrm{SC}(S)$ and $(a, b) \in \sigma$ then $(a)_\sigma = (b)_\sigma$. Since $(b)_\sigma$ is Archimedean, $a^n \leq ubv$ for some positive integer $n \in N$ and $u, v \in (b)_\sigma$. Clearly, $N(a^n) \subset N(a)$, thereby $ubv \in N(a)$ and so $b \in N(a)$, since $N(a)$ is a filter. This shows that $N(b) \subseteq N(a)$. Similarly, we can prove that $N(a) \subseteq N(b)$. Thus $N(a) = N(b)$, that is, $(a, b) \in \mathcal{N}$. Hence, $\sigma \subseteq \mathcal{N}$, in otherwords, $\mathcal{N}$ is the greatest congruence among all the semilattice congruences defined on $S$.

Finally, we demonstrate that the semilattice decomposition of a pseudo-commutative po-semigroup need not be unique.

**Example 4.7** Let $S = \{a, b, c, d\}$ be a set with Cayley table and Hasse diagram shown below:

| $\cdot$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $b$ | $b$ | $c$ | $c$ |
| $b$ | $b$ | $b$ | $c$ | $c$ |
| $c$ | $c$ | $c$ | $c$ | $c$ |
| $d$ | $c$ | $c$ | $c$ | $c$ |




Then $S$ is a po-semigroup (One can use the method adopted by Keyayopulu[3] to verify the details). Since all principal ordered filters of the above semigroup $S$ are strictly simple, by Proposition 3.9, $S$ is pseudo-commutative as well as 2-cyclic-commutative.

Now, let $\mathcal{N} = \{(x, y) \in S \times S | N(x) = N(y)\} = S \times S$, and $\eta = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (c, d), (d, c)\}$. Then, we can verify that $\mathcal{N}$ and $\eta$ are semilattice congruences on $S$. Furthermore, for any $\sigma \in \mathrm{SC}(S)$, we have either $\sigma = \mathcal{N}$ or $\eta$. Thus $\mathrm{SC}(S) = \{\mathcal{N}, \eta\}$. Clearly, $\eta$ does not satisfy the condition in Lemma 4.4, so we have $\eta \not\subseteq \mathcal{N}$. Considering the $\eta$-classes. We can easily find that $(a)_\eta = (b)_\eta = \{a, b\}; (c)_\eta = (d)_\eta = \{c, d\}$. Since $a^2 = b$ and $b^2 = b$, we have $(a)_\eta$ is Archimedean. Similary, $d^2 = c, c^2 = c$, we have $(c)_\eta$ is Archimedean. Thus, apart from $\mathcal{N}$, the congruence $\eta$ is also a semilattice congruence on $S$. This example shows that the decomposition of a po-semigroup $S$ into a semilattice of Archimedean po-semigroups is not necessarily unique.

We would like to mention here that there are pseudo-commutative po-semigroups on which the semilattice decomposition into Archimedean po-semigroups can be unique. Example 3.11, for instance, is such an example. In fact, in Example 3.11, we can observe that $\mathrm{SC}(S) = \{\mathcal{N}, \sigma\}$, where $\mathcal{N} = \{(a, b) | N(a) = N(b)\} = \{(a, a), (b, b), (c, c), (d, d), (a, c), (c, a), (b, d), (d, b)\}$ and $\sigma = S \times S$. However, it is clear that $(b)_\sigma$ is not Archimedean for $(b)_\sigma = S$ and $a^n \not\leq xby$ for all $n \in N = \{1, 2, \cdots\}$, where $x, y \in (b)_\sigma$. Thus, in this example,

$S$ can only be decomposed uniquely into semilattice of Archimedean po-semigroups with respect to the semilattice congruence $\mathcal{N}$.

**Note** (i) In Example 4.7, the semilattice congruence $\mathcal{N}$ is not the least semilattice congruence on the po-semigroup $S$. This answers a question posed by Kehayopulu in [4].

(ii) Some of the results on semilattice decomposition of weakly commutative semigroups were annouced by Kehayopulu and Tsingelis in AMS Abstracts (15)(3), 1994.

**Acknowledgement** The authors would like to thank Professor Niovi Kehayopulu for her valuable suggestions and useful comments contributed to this paper.

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# 几 类 亚 交 换 序 半 群

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## 摘 要

本文研究几类亚交换序半群的性质，并将具有单位元的亚交换序半群的一些结果扩张到这几类亚交换序半群上，使得这些结果更加细化．其中主要证明了以下定理：伪交换序半群可以分解成阿基米德序半群的半格．并且，一般来说，这种分解不是唯一的．