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The Automorphism Groups of Complete Lie Algebras with Commutative Nilpotent Radical *

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Abstract: The automorphism groups of finite-dimensional complete Lie algebras with commutative nilpotent radical over complex field $\mathcal{C}$ are given.

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1. Introduction

A Lie algebra is called complete Lie algebra if its centre is zero and its all derivations are inner. A complete Lie algebra is called simple complete Lie algebra if it has no non-trivial complete ideals.

Let $\mathcal{L}$ be a finite-dimensional Lie algebra over complex field $\mathbf{C}$. Then $\mathcal{L}$ has the Levi decomposition:

$$\mathcal{L} = \mathcal{S} + \mathcal{R},$$

where $\mathcal{S}$ is a maximal semisimple Lie subalgebra of $\mathcal{L}$, called a Levi subalgebra of $\mathcal{L}$; $\mathcal{R}$ is the maximal solvable ideal of $\mathcal{L}$, called the radical of $\mathcal{L}$. The ideal $\mathcal{N} = [\mathcal{L}, \mathcal{R}]$ is called the nilpotent radical of $\mathcal{L}$.

Since $\mathcal{S}$ is semisimple, $\mathcal{R}$ can be viewed as $\mathcal{S}$-module. Let $\mathcal{R}_n$ be the direct sum of non-trivial irreducible submodules, $\mathcal{R}_0$ the direct sum of one dimensional submodules. [1] proved that if $\mathcal{L}$ is a complete Lie algebra with commutative nilpotent radical, then $\mathcal{L}$ is a direct sum of two complete ideals:

$$\mathcal{L} = (\mathcal{S} + C(\mathcal{R}_0) + \mathcal{R}_n) \oplus C_{\mathcal{R}_0}(\mathcal{R}_n)$$

and

$$C_{\mathcal{R}_0}(\mathcal{R}_n) = \{x \in \mathcal{R}_0 | \mathrm{adx}(\mathcal{R}_n) = 0\}$$

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is a direct sum of 2-dimensional complete ideals. [2] proved that $\mathcal{R}_n$ can be decomposed into direct sum of irreducible submodules $V_1, V_1, \cdots, V_m$ such that

$$C(\mathcal{R}_0) = \mathbf{C}I_1 + \mathbf{C}I_2 + \cdots + \mathbf{C}I_m, \quad \mathrm{ad}I_i|_{V_j} = \delta_{ij} id, i, j = 1, 2, \cdots, m.$$

Therefore if $\mathcal{L}$ is a solvable complete Lie algebra with commutative nilpotent radical, then $\mathcal{L}$ is simple complete if and only if $\mathcal{L}$ is a 2-dimensional complete Lie algebra. In this case, the automorphism group of $\mathcal{L}$ is isomorphic to the matrix group

$$\left\{ \left[\begin{array}{cc} 1 & 0 \\ a_1 & a_2 \end{array} \right] \middle| a_1, a_2 \in \mathbf{C}, a_2 \neq 0 \right\}.$$

Therefore throught out the paper we assume that $\mathcal{L}$ is a finite-dimensional non-solvable simple complete Lie algebra with commutative nilpotent radical over $\mathbf{C}, \mathcal{S}_0$ is a Levi subalgebra of $\mathcal{L}$ and $\mathcal{S}_0$ is simple.

2. The automorphism group Aut$\mathcal{L}$ of $\mathcal{L}$

We know every finite-dimensional $\mathcal{S}_0$-module is a highest weight module, so we can assume that

$$\mathcal{L} = \mathcal{S}_0 + \sum_{i=1}^{m}\sum_{j=1}^{n_i} \mathbf{C}I_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n_i} V_{ij}, \tag{1}$$

where $V_{ij}(j = 1, 2, \cdots, n_i)$ are irreducible highest weight modules with highest weight $\lambda_i, i = 1, 2, \cdots, m$, and

$$[\mathcal{S}, I_{ij}] = 0, [V_{ij}, V_{kl}] = 0, \tag{2}$$

$$\mathrm{ad}I_{ij}|_{V_{kl}} = \delta_{ik}\delta_{jl} id, j = 1, \cdots, n_i, i = 1, \cdots, m, \ell = 1, \cdots, k_i, k = 1, 2, \cdots, m. \tag{3}$$

It is clear that the nilpotent radical of $\mathcal{L}$ is $\mathcal{N} = \sum_{i=1}^{m}\sum_{j=1}^{n_i} V_{ij}$, the radical of $\mathcal{L}$ is

$$\mathcal{R} = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \mathbf{C}I_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n_i} V_{ij}.$$

Lemma 1[1] *Let G_1 be a subgroup of* Aut$\mathcal{L}$ *gererated by* $\{\mathrm{expadx}|x \in \mathcal{N}\}$. *Let $\mathcal{S}_1$ be any Levi subalgebra of $\mathcal{L}$. Then there exists $\sigma \in G_1$ such that $\sigma\mathcal{S}_0 = \mathcal{S}_1$.*

Lemma 2[3] *Let* Int$\mathcal{S}_0$ *be the inner automorphism group of $\mathcal{S}_0$ generated by* $\{\mathrm{expadx}|x \in \mathcal{S}_0$ *is ad-nilpotent*$\}$, Γ_0 *the graph-automorphism group of $\mathcal{S}_0$. Then* Aut$\mathcal{S}_0$ *is the semidirect product of* Int$\mathcal{S}_0$ *and* Γ_0.

Set

$$G_0 = \{\sigma \in \mathrm{Aut}\mathcal{L} | \sigma|_{\mathcal{S}_0} = \mathrm{id}\}, \tag{4}$$

$$W = \{\sum_{i=1}^{m}\sum_{j=1}^{n_j} \mathbf{C}I_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n_j} \mathbf{C}v_{ij} | v_{ij} \text{ is a highest weight vector of } V_{ij}$$

$$\text{associated with highest weight } \lambda_i\}, \tag{5}$$

$$W_i = \{\sum_{j=1}^{n_i} a_{ij} v_{ij} | a_{ij} \in \mathbf{C}\}, i = 1, 2, \cdots, m, \tag{6}$$

$$N_0 = \{\sigma|_W | \sigma \in G_0\}. \tag{7}$$

Lemma 3 *Let $\sigma \in G_0$ and $V_\lambda \subseteq \mathcal{N}$ be the weight vector space associated with weight λ. Then $\sigma V_\lambda \subseteq V_\lambda$ and $\sigma W_i \subseteq W_i, i = 1, 2, \cdots, m$.*

Proof Since $\sigma|_{\mathcal{S}_0} = \mathrm{id}$, for any $x \in \mathcal{S}, v \in \mathcal{N}$ we have

$$\sigma[x, v] = [x, \sigma v]. \tag{8}$$

It is easy to prove the lemma by (8). □

Lemma 4 *The group G_0 is isomorphic to N_0.*

Proof Define $f : G_0 \to N_0$ by $\sigma \to \sigma|_W$. It is clear that f is a homomorphic mapping. Let $\sigma_1, \sigma_2 \in G_0$ be such that $\sigma_1|_W = \sigma_2|_W$. Then

$$\sigma_1(v_{ij}) = \sigma_2(v_{ij}), j = 1, \cdots, n_i, i = 1, \cdots, m.$$

As V_{ij} is a highest weight module, for any $v \in V_{ij}, v$ has the following form:

$$v = [x_1, [x_2, \cdots, [x_n, v_{ij}] \cdots],$$

where $x_1, x_2, \cdots, x_n \in \mathcal{S}_0$. By $\sigma_1|_{\mathcal{S}_0} = \sigma_2|_{\mathcal{S}_0} = \mathrm{id}$, we have

$$\sigma_1 v = [x_1, [x_2, \cdots, [x_n, \sigma_1 v] \cdots], \ \sigma_2 v = [x_1, [x_2, \cdots, [x_n, \sigma_2 v] \cdots].$$

Therefore $\sigma_1 v = \sigma_2 v$. □

Lemma 5 *Let $\sigma \in G_0$. Then*

$$\sigma|_{\mathcal{S}_0} = \mathrm{id}, \sigma I_{ik} = I_{ii_k}, \sigma v_{ik} = c_{ik} v_{ii_k}, k = 1, 2, \cdots, n_i, i = 1, \cdots, m, \tag{9}$$

$$\sigma([x_1, \cdots, [x_n, v_{ik}] \cdots]) = [x_1, [x_2, \cdots, [x_n, \sigma v_{ik}] \cdots], \tag{10}$$

where $i_1, i_2, \cdots, i_{n_i}$ is a range of $1, 2, \cdots, n_i, c_{ik} \in \mathbf{C}^* = \mathbf{C} \backslash \{0\}, x_1, x_1, \cdots, x_n \in \mathcal{S}_0, n \in \mathbf{Z}$ and $n \geq 1$.

Proof By the fact that $[\mathcal{S}_0, I_{ij}] = 0$, we have

$$\sigma[\mathcal{S}_0, I_{ij}] = [\mathcal{S}_0, \sigma I_{ij}] = 0. \tag{11}$$

Since $\sigma I_{ij} \subseteq \mathcal{R}$ and $V_{ij} (j = 1, \cdots, n_i, i = 1, \cdots, m)$ are non-trivial irreducible $\mathcal{S}_0$-modules, by (11) we have

$$\sigma I_{ij} \subseteq \sum_{i=1}^{m} \sum_{j=1}^{n_j} \mathbf{C} I_{ij}.$$

By the fact that $\sigma W_i \subseteq W_i (i = 1, 2, \cdots, m)$ we know that $\sigma I_{ij} \subseteq \sum_{k=1}^{n_i} \mathbf{C} I_{ik} = U_i$.

Let $A_i = (a_{kl}^{(i)})_{k,l=1}^{n_i}$ be the matrix of $\sigma|_{U_i}$ relative to the base $\{I_{i1}, I_{i2}, \cdots, I_{in_i}\}$ of $U_i, B_i = (b_{kl}^{(i)})_{k,l=1}^{n_i}$ the matrix of $\sigma|_{W_i}$ relative to the base $\{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$ of W_i. Since

$$[I_{ik}, v_{ij}] = \delta_{kj} v_{ij}, \sigma[I_{ik}, v_{ij}] = [\sigma I_{kj}, \sigma v_{ij}],$$

we have

$$B_i E_{jj} = (a_{1j}^{(i)} E_{11} + a_{2j}^{(i)} E_{22} + \cdots + a_{n_i j}^{(i)} E_{n_i n_i}) B_i, \quad j = 1, 2, \cdots, n_i, \tag{12}$$

where E_{jj} denotes the $n_i \times n_i$ matrix which is 1 in the j, j entry and 0 everywhere else. As $\sigma \in \mathrm{Aut}\mathcal{L}$, A_i and B_i are reversible matrices. Therefore E_{jj} is similar to $\sum_{k=1}^{n_i} a_{kj}^{(i)} E_{kk}$, so there exists only one non-zero element in $\{a_{1j}^{(i)}, a_{2j}^{(j)}, \cdots, a_{n_i j}^{(i)}\}$. Assume that $a_{kj}^{(i)}$ is non-zero, then $a_{kj}^{(i)} = 1$. By (12) we can deduce that only $b_{kj}^{(i)}$ is non-zero in $\{b_{1j}, b_{2j}, \cdots, b_{n_i j}\}$. Therefore (9) is true.

Theorem 1 *G_0 consists of the transformations of $\mathcal{L}$ satisfing (9) and (10).*

Proof Let σ_0 be a transformation of W such that

$$\sigma_0(I_{ik}) = I_{ii_k}, \sigma_0(v_{ik}) = c_{ik} v_{ii_k}, k = 1, 2, \cdots, n_i, i = 1, 2, \cdots, m,$$

where $i_1, i_2, \cdots, i_{n_i}$ is a range of $1, 2, \cdots, n_i, c_{ik} \in \mathbf{C}^* = \mathbf{C} \backslash \{0\}$. It is clear that σ_0 is an automorphism of Lie subalgebra W of $\mathcal{L}$. Extend σ_0 to the transformation σ of $\mathcal{L}$ by

$$\sigma|_W = \sigma_0, \sigma|_{\mathcal{S}_0} = \mathrm{id}, \ \sigma([x_1, [x_2, \cdots, [x_n, v_{ik}] \cdots]) = [x_1, [x_2, \cdots, [x_n, \sigma_0 v_{ik}] \cdots],$$

where $x_1, x_2, \cdots, x_n \in \mathcal{S}_0$. Since $V_{i1}, V_{i2}, \cdots, V_{in_i}$ are isomorphic each other, it is easy to prove that $\sigma \in \mathrm{Aut}\mathcal{L}$. By Lemma 4 and Lemma 5 the Theorem holds. □

Theorem 2 *If $\mathcal{S}_0$ is not $A_l (l > 1), D_l (l \geq 4)$ or E_6, then $\mathrm{Aut}\mathcal{L} = (\mathrm{Int}\mathcal{L}) G_0$, where $\mathrm{Int}\mathcal{L}$ is the inner automorphism group of $\mathcal{L}$ generated by $\{\mathrm{expadx} | x \in \mathcal{L}$ is adnilpotent$\}$.*

Proof Let $\sigma \in \mathrm{Aut}\mathcal{L}$. Then $\sigma\mathcal{S}_0$ is a Levi subalgebra of $\mathcal{L}$. By Lemma 1, there exists $\sigma_0 \in \mathrm{Int}\mathcal{L}$ such that $\sigma\mathcal{S}_0 = \sigma_0\mathcal{S}_0$. Therefore $\sigma_0^{-1}\sigma|_{\mathcal{S}_0} \in \mathrm{Aut}\mathcal{S}_0$. By Lemma 2, $\sigma_0^{-1}\sigma|_{\mathcal{S}_0} \in \mathrm{Int}\mathcal{S}_0$, so there exists $\tau \in \mathrm{Int}\mathcal{L}$ such that $\tau|_{\mathcal{S}_0} = \sigma_0^{-1}\sigma|_{\mathcal{S}_0}$. Therefore $\tau^{-1}\sigma_0^{-1}\sigma \in G_0$. We deduce that $\sigma \in (\mathrm{Int}\mathcal{L})G_0$. On the other hand, we know that $(\mathrm{Int}\mathcal{L})G_0 \subseteq \mathrm{Aut}\mathcal{L}$. The theorem holds. □

Lemma 6 *Let $\mathcal{G}$ be a Cartan subalgebra of $\mathcal{S}_0, \Delta^\vee = \{\alpha_1^\vee, \cdots, \alpha_n^\vee\}$ the simple coroot system of $\mathcal{S}_0, \tau_0$ a graph-automorphism of $\mathcal{S}_0$. Let $\mu_{ij} \in \mathcal{G}^*$ be such that*

$$\mu_{ij}(\alpha_k^\vee) = \lambda_i(\tau_0(\alpha_k^\vee))(j = 1, 2, \cdots, n_i, i = 1, 2, \cdots, m, k = 1, \cdots, n).$$

Then τ_0 can be extended to an automorphism of $\mathcal{L}$ if and only if

$$\{\mu_{ij} | j = 1, 2, \cdots, n_i, i = 1, 2, \cdots, m\} = \{\lambda_{ij} = \lambda_i | j = 1, 2, \cdots, n_i, i = 1, 2, \cdots, m\}.$$

Proof Let $\{e_i, f_i | i = 1, 2, \cdots, n\}$ be the Chevalley generators of $\mathcal{S}_0$. Assume that

$$\tau_0(\alpha_k^\vee) = \alpha_{ik}^\vee, k = 1, 2, \cdots, n.$$

If τ_0 can be extended to an automorphism τ of $\mathcal{L}$, then τv_{ij} is a highest weight vector associated with highest weight μ_{ij}. So

$$\{\mu_{ij}|j=1,2,\cdots,n_i,i=1,2,\cdots,m\}=\{\lambda_{ij}=\lambda_i|j=1,2,\cdots,n_i,i=1,2,\cdots,m\}.$$

Inversely, assume that $\{\mu_{ij}|j=1,2,\cdots,n_i,i=1,2,\cdots,m\}=\{\lambda_{ij}=\lambda_i|j=1,2,\cdots,n_i,i$ $1,2,\cdots,m\}$. Then for any module V_{kl}, there is a responding highest weight module $V_{i_k i_l}$ in $\{V_{ij}|j=1,2,\cdots,n_i,i=1,2,\cdots,m\}$ associated with highest weight $\mu_{kl}=\lambda_{i_k i_l}$. Define

$$\tau|_{\mathcal{S}_0}=\tau_0,\tau(v_{kl})=v_{i_k i_l},$$

$$\tau([x_1,[x_2,\cdots,[x_s,v_{kl}]\cdots])=[\tau_0 x_1,[\tau_0 x_2,\cdots,[\tau_0 x_s,\tau v_{kl}]\cdots],\tau(I_{kl})=I_{i_k i_l},$$

where $x_1,x_2,\cdots,x_s\in\mathcal{S}_0$. It is easy to prove that $\tau\in\mathrm{Aut}\mathcal{L}$. □

Theorem 3 *If $\mathcal{S}_0$ is $A_l(l>1),D_l(l\geq 4)$ or E_6. Then*

(1) $\mathrm{Aut}\mathcal{L}=(\mathrm{Int}\mathcal{L})G_0\Gamma_0,\Gamma_0$ *consists of the automorphisms of $\mathcal{L}$ introduced in the proof of Lemma 6.*

(2) $(\mathrm{Int}\mathcal{L})G_0$ *is a normal subgroup of* $\mathrm{Aut}\mathcal{L}$ *and if $\mathcal{S}_0$ is $A_l(l>1),D_l(l>4)$ or E_6, then* $[\mathrm{Aut}\mathcal{L}:(\mathrm{Int}\mathcal{L})G_0]\leq 2$, *if $\mathcal{S}_0$ is D_4, then* $[\mathrm{Aut}\mathcal{L}:(\mathrm{Int}\mathcal{L})G_0]\leq 6$.

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具有交换幂零根基的完备李代数的自同构群

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摘 要: 给出了复数域上具有交换幂零根基的完备李代数的自同构群.