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Characterization of Connected Graphs with Maximum Domination Number *

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Abstract: Let G be a connected graph of order p, and let $\gamma(G)$ denote the domination number of G. Clearly, $\gamma(G) \leq [p/2]$. The aim of this paper is to characterize the graphs G that reaches this upper bound. The main results are as follows: (1) when p is even, $\gamma(G) = \frac{p}{2}$ if and only if either $G \cong C_4$ or G is the crown of a connected graph with $\frac{p}{2}$ vertices; (2) when p is odd, $\gamma(G) = \frac{p-1}{2}$ if and only if every spanning tree of G is one of the two classes of trees shown in Theorem 3.1.

Key words: connected graph; crown; domination number; domination critical graph.

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1. Introduction

We use Bondy and Murty[1] for terminologies and notations not defined here and consider simple graphs only.

Let $G = (V, E)$ be a graph, $V = V(G)$ and $E = E(G)$ denote vertex and edge-set of G, resp. If $v \in V(G), N(v)$ denotes the open neighborhood, $d(v) = |N(v)|$ is the degree of $v, N[v] = N(v) \cup \{v\}$. If $S \subset V(G), G[S]$ is the subgraph of G induced by $S, G - S$ denotes the one induced by $V(G) - S$. A vertex x is often identified with $\{x\}, N(S) = \cup_{v \in S} N(v)$. If $x, y \in V(G), d(x, y)$ denotes the distance between x and y. P_m, C_m and K_m are the path, cycle and complete graph of order m, resp. If x is a real number, then $[x]$ denotes the greatest integer not larger than x.

A dominating set D for a graph $G = (V, E)$ is a subset of V such for all $v \in V - D$, there exists some $u \in D$ for which $uv \in E$. The domination number of G is the size of its smallest dominating set (s) and is denoted by $\gamma(G)$.

Let H be a graph, A graph G is said to be the crown of H if G can be obtained by adding a pendant-edge at each vertex of H, and the graph G (crown of H) is denoted by H^*. Clearly, for all graphs $H, |V(H^*)| = 2|V(H)|$ and $|E(H^*)| = |E(H)| + |V(H)|$.

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The following lemma is due to Ore[2]

Lemma 1.1 *If a graph G with p vertices has no isolated vertices then $\gamma(G) \leq [\frac{p}{2}]$.*

A graph G without isolated vertices attains this upper bound if and only if it has at most one odd component, and its every component attains its corresponding bound. Thus, we need only consider connected graphs.

In this paper, we characterize the connected extremal graphs attaining that upper bound. In Section 2, considering the extremal graphs with $2n$ vertices, we prove the following result: Let G be a connected graph with $2n$ vertices, then $\gamma(G) = n$ if and only if either $G \cong C_4$ or G is the crown of some connected graph with n vertices. In order to characterize the connected extremal graphs of order $2n+1$, in Section 3, we construct the two classes of trees, which are written as $\tilde{T}_1$ and $\tilde{T}_2$. (see Theorem 3.1), we prove the result: Let T be a tree of order $2n+1$, then $\gamma(T) = n$ if and only if $T \in \tilde{T}_1 \cup \tilde{T}_2$. Furthermore, if G is a connected graph with $2n+1$ vertices, then $\gamma(G) = n$ if and only if every spanning tree of G is in $\tilde{T}_1$ or $\tilde{T}_2$.

In order to obtain our main results, let us list some useful lemmas as follows:

Lemma 1.2 *If H is a spanning subgraph of G, then $\gamma(H) \geq \gamma(G)$.*

Lemma 1.3 *Let G be a graph, $S \subseteq V(G)$, then $\gamma(G) \leq \gamma(G[S]) + \gamma(G-S)$.*

Lemma 1.4 *For all graphs $G, \gamma(G^*) = |V(G)|$.*

2. The extremal graphs with $2n$ vertices

In this section, we establish our theorem to characterize the connected extremal graphs of order $2n$.

Theorem 2.1 *If G is a connected graph of order $2n$, then $\gamma(G) = n$ if and only if $G \cong C_4$ or G is the crown of some connected graph with n vertices.*

Proof Sufficiency: It is obvious from Lemma 1.4 and the fact $\gamma(C_4) = 2$.

Necessity: Let G be a connected graph of order $2n$, and $\gamma(G) = n$.

Let $D_1 = \{v \in V(G) | d(v) = 1\}. D_2 = N(D_1)$.

Claim 1 For all $x, y \in D_1 (x \neq y)$, we have $N(x) \cap N(y) = \emptyset$.

Assume, to the contrary, that $N(x) \cap N(y) \neq \emptyset$ and $v \in N(x) \cap N(y)$. Let $S_1 = N(v) \cap D_1, S = S_1 \cup \{v\}$. It is obvious that $|S| \geq 3$ since $x, y \in S_1$ and $v \notin D_1. \gamma(G[S]) = 1$ for it is a star. Clearly, there is no any isolated vertices in $G-S$. By Lemma 1.1 and 1.3, we have $\gamma(G) \leq \gamma(G[S]) + \gamma(G-S) \leq 1 + [\frac{2n-3}{2}] = n-1$, this is a contradiction.

Claim 2 If there exists a vertex $v \in V(G) - D_1$ such that $N(v) \cap D_1 = \emptyset$, then $D_1 = \emptyset$ (namely, $\delta(G) \geq 2$).

Assume, to the contrary, that $D_1 \neq \emptyset$. We choose such a vertex $v \in V(G) - D_1$ and $N(v) \cap D_1 = \emptyset$ that there is at least one vertex $u \in N(v)$ and $N(u) \cap D_1 \neq \emptyset$.

Let $N(u) \cap D_1 = \{u_1, u_2, \cdots, u_t\} (t \geq 1)$

$$S_1 = \{v, u, u_1, u_2, \cdots, u_t\}, S_2 = \{x \in V(G) | N(x) = \{v, u\}\},$$

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$$S = S_1 \cup S_2, |S| \geq t + 2 \geq 3.$$

Since $\{u\}$ is a dominating set of $G[S]$ (namely, $\gamma(G[S]) = 1$), and there is no any isolated vertices in $G - S$, by Lemma 1.1 and 1.3, we get $\gamma(G) \leq \gamma(G[S]) + \gamma(G - S) \leq 1 + [\frac{2n-3}{2}] = n - 1$, a contradiction.

Summing up the above two claims, we have proved that every non-pendant vertex of G is adjacent to exactly one pendant vertex if $D_1 \neq \emptyset$. Namely, G is the crown of the graph $G - D_1$.

Claim 3 If $D_1 = \emptyset(\delta(G) \geq 2)$, then $n = 2$ and $G \cong C_4$.

Choose such a vertex $v \in V(G)$ that $d(v) = \delta(G) \geq 2$.

Let $S_1 = \{x \in V(G) | N(x) = N(v)\}$, note that $S_1 \neq \emptyset$ since $v \in S_1 . S = S_1 \cup N(v), G[S]$ is a connected graph, and there is no any isolated vertices in $G - S$, by Lemma 1.1 and 1.3, we have

$$n = \gamma(G) \leq \gamma(G[S]) + \gamma(G - S) \leq [\frac{|S|}{2}] + [\frac{2n - |S|}{2}] \leq n.$$

This implies $|S| = 2t$ is even and $\gamma(G[S]) = t$.

Let $G_1 = G[S], G_2 = G - S$.

Next we'll prove that $G_1 \cong C_4$ and $G_2 = \emptyset$.

It is obvious that $E(G_1[S_1]) = \emptyset$. And further, $E(G_1[N(v)]) = \emptyset$ (otherwise, there exists $v_1 \in N(v)$ such that v_1 is adjacent to some vertices in $N(v)$. Obviously, S_1 and $N(v) - \{v_1\}$ are two disjoint dominating sets of G_1, and $|S_1| + |N(v) - \{v_1\}| = |S| - 1 = 2t - 1$, which contradicts $\gamma(G_1) = \frac{|S|}{2} = t$), note that the definition of S_1, we know G_1 is a complete bipartite graph. Hence, $\gamma(G_1) = t = 2$ and $|V(G_1)| = 2t = 4$. Which implies $G_1 \cong K_{2,2} = C_4$.

Now we prove $G = G_1 = C_4$ (nemaly, $G_2 = \emptyset$).

Assume, $G_2 \neq \emptyset$, there exists $w \in V(G_2)$ such that w is adjacent to at least one of v_1 and v_2 in G_1, where $\{v_1, v_2\} = N(v), V(G_1) = \{v, v_1, v', v_2\}$ and $d(v') = 2$. Without loss of generality we may suppose $wv_1 \in E(G)$.

Let $S' = \{v, v', v_1, v_2, w\}, S'' = \{x \in V(G_2) | N(x) \subseteq S'\}$

$S_0 = S' \cup S'', G_3 = G[S_0], |S_0| \geq 5$, note that $\delta(G) \geq 2$, it is easy to see that $\{w, v_2\}$ is a dominating set of G_3, this is, $\gamma(G_3) \leq 2$. There is no any isolated vertices in $G - S_0$, by Lemma 1.1 and 1.3, we have $\gamma(G) \leq \gamma(G_3) + \gamma(G - S_0) \leq 2 + [\frac{2n-5}{2}] = n - 1$, a contradiction. We have finished the proof of Theorem 2.1. □

The following two corollaries are immediate from Theorem 2.1.

Corollary 2.2 *If G is a graph of order $2n, \delta(G) \geq 1$, then $\gamma(G) = n$ if and only if every component of G is C_4 or the crown of some connected graph.*

A graph G is said to be k-domination critical if $\gamma(G) = k$ and for every edge $e \in E(\overline{G}), \gamma(G + e) < \gamma(G)$.

Corollary 2.3 *If G is a connected graph with $2n$ vertices, then G is n-domination critical if and only if*

$$G \cong \begin{cases} C_4, & \text{when } n = 2, \\ K_n^*, & \text{when } n \geq 3. \end{cases}$$

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3. The extremal graphs with $2n+1$ vertices

In this section, we consider the connected extremal graphs of order $2n+1$. First, we construct two classes of trees, which are written as $\tilde{T}_1$ and $\tilde{T}_2$. We define

$$\tilde{T}_1 = \{T^* - v | T \text{ is a tree of order } n+1 \text{ and } v \in V(T^*) - V(T)\}$$

Namely, $T \in \tilde{T}_1$ if and only if T can be obtained by deleting some pendant vertex from the crown of some tree with $n+1$ vertices.

Let $\tilde{T}_2 = \{T | T \text{ is such a tree of order } 2n+1 \text{ that it can be constructed by Fig.1}\}$

In Fig.1, $d(u) = 2, d(u_1) \geq 2, d(u_2) \geq 2. T_j^*$ is the crown of some tree $T_j. v_j$ is adjacent to exactly one pendant vertex v_j' in $T_j^* (1 \leq j \leq s+t)$. Obviously, $u_i (i=1,2)$ is not adjacent to any pendant vertices, $s \geq 1$ and $t \geq 1. \sum_{j=1}^{s+t} |V(T_j)| = n-1$.

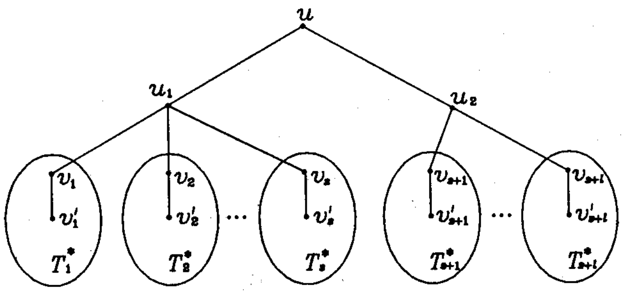

Fig.1

Theorem 3.1 *If T is a tree of order $2n+1$, then $\gamma(T) = n$ if and only if $T \in \tilde{T}_1 \cup \tilde{T}_2$.*

Proof Sufficiency: (1) $T \in \tilde{T}_1$; then $T = T_0^* - v_0$ for some tree T_0 of order $n+1$, v_0 is some pendant vertex in T_0^*. By Theorem 2.1, $\gamma(T_0^*) = n+1$. Let D be the smallest dominating set of T. Clearly, $D \cup \{v_0\}$ is a dominating set of T_0^*, it implies

$$\gamma(T) + 1 \geq \gamma(T_0^*) = n+1, \gamma(T) \geq n.$$

On the other hand, $\gamma(T) \leq [\frac{|V(T)|}{2}] = n$, thus $\gamma(T) = n$.

(2) $T \in \tilde{T}_2$: T can be constructed in Fig.1, since $T_1^* \cup T_2^* \cup \cdots \cup T_{s+t}^* \cup P_3$ is a spanning subgraph of T, by Lemma 1.2 and Theorem 2.1, we have

$$\gamma(T) \leq \gamma(T_1^* \cup T_2^* \cup \cdots \cup T_{s+t}^* \cup P_3) = \sum_{j=1}^{s+t} |V(T_j)| + 1 = n.$$

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On the other hand, let D be the smallest dominating set of T. Clearly, for every pendant vertex $v \in V(T), N[u] \cap D \neq \emptyset, N[v] \cap D \neq \emptyset, N[u] \cap N[v] = \emptyset$, if v' is another pendant vertex, $N[v'] \cap N[v] = \emptyset$. Which imply $|D| \geq n$, thus $\gamma(T) = n$.

Necessity: T is a tree with $2n+1$ vertices and $\gamma(T) = n$. Let $D_1 = \{v \in V(T) | d(v) = 1\}, D_2 = V(T) - D_1$. Note that $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$ (otherwise, $T = K_2$, this contradicts $|V(T)| = 2n+1$).

Case 1 $\forall x \in D_2, N(x) \cap D_1 \neq \emptyset$.

In this case, $|D_1| \geq |D_2|$. It is easy to see that D_2 is a dominating set of T, thus $|D_2| \geq n$, note that $|D_1| + |D_2| = 2n+1$, we have $|D_1| = n+1$ and $|D_2| = n$. Hence, there exists exactly one vertex $x_0 \in D_2$ such that $|N(x_0) \cap D_1| = 2$, and when $x \neq x_0, x \in D_2, |N(x) \cap D_1| = 1$. Which imply $T \in \tilde{T}_1$.

Case 2 There exist some vertices $x \in D_2$ such that $N(x) \cap D_1 = \emptyset$.

Subcase 2.1 There is exactly one vertex $v \in D_2$ such that $N(v) \cap D_1 = \emptyset$.

In this subcase, we'll prove that for every $x \in D_2 (x \neq v), |N(x) \cap D_1| = 1$ (It implies $T \in \tilde{T}_1$).

Assume, to the contrary, that there exists $v' \neq v, v' \in D_2$ such that $|N(v') \cap D_1| \geq 2$. Let $P_{t+1} = (vv_1v_2 \cdots v_{t-1}v')$ be the shortest path joining v and v', where $t = d(v, v')$.

Let $S_1 = \{x \in V(T) | d(x) = 1 \text{ and } N(x) \subseteq V(P_{t+1})\}$. Obviously, $|S_1| \geq t+1$, let $S = S_1 \cup V(P_{t+1}), |S| \geq 2t+2. \{v_1, v_2, \cdots, v_{t-1}, v'\}$ is a dominating set of $T[S]$, it implies $\gamma(T[S]) \leq t. T - S$ has no any isolated vertices, by Lemma 1.1 and 1.3 we get

$$\gamma(T) \leq \gamma(T[S]) + \gamma(T-S) \leq t + [\frac{(2n+1)-(2t+2)}{2}] = n-1$$

a contradiction.

Subcase 2.2 D_2 has at least two vertices x and y such that $N(x) \cap D_1 = \emptyset$ and $N(y) \cap D_1 = \emptyset$.

In this subcase, $D_2 - \{x, y\}$ is a dominating set of T. It implies $|D_2| \geq n+2$ and hence $|D_1| \leq n-1$, thus, D_1 is not a dominating set of T, namely, there exists a vertex $u \in D_2$ such that $N[u] \cap N(D_1) = \emptyset$. Furthermore, $d(u) = 2$ (otherwise, $d(u) \geq 3. T - N[u]$ has no any isolated vertices, by Lemma 1.1 and 1.3,

$$\gamma(T) \leq \gamma(T - N[u]) + \gamma(T[N[u]]) \leq [\frac{2n+1-4}{2}] + 1 = n-1,$$

a contradiction). Let $N(u) = \{u_1, u_2\}$, note that $|V(T - N[u])| = 2n-2$, thus,

$$n = \gamma(T) \leq \gamma(T[N[u]]) + \gamma(T - N[u]) \leq 1 + (n-1) = n.$$

It implies $\gamma(T - N[u]) = n-1$, by Theorem 2.1, every component of $T - N[u]$ is the crown of a tree, all those components are written as $T_1^*, T_2^*, \cdots, T_{s+t}^*$.

For every $j (1 \leq j \leq s+t), T_j^*$ has exactly one vertex v_j adjoining to one of u_1 and u_2. Without loss of generality we may suppose that v_j is adjacent to u_1.

Next we prove that v_j is adjacent to (exactly) one pendant vertex in T_j^*.

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Assume, to the contrary, that $N(v_j) \cap D_1 = \emptyset$, note that T_j^* is the crown of T_j, there is a vertex $v' \in D_1$ such that $d(v_j, v') = 2$. Let $N(v') = \{v''\}, P_6 = (v'v''v_ju_1uu_2)$ be a path in T, it is easy to see that $T - V(P_6)$ has no any isolated vertices. By Lemma 1.1 and 1.3, we have

$$\gamma(T) \leq \gamma(P_6) + \gamma(T - V(P_6)) \leq 2 + [\frac{2n+1-6}{2}] = n - 1,$$

this is a contradiction. Hence, in subcase 2.2, we have proved $T \in \tilde{T}_2$.

We have finished the proof of Theorem 3.1. □

Theorem 3.2 *If G is a connected graph of order $2n+1$, then $\gamma(G) = n$ if and only if every spanning tree of G is in $\tilde{T}_1 \cup \tilde{T}_2$.*

Proof Necessity: It is immediately from Lemma 1.2 and Theorem 3.1.

Sufficiency: Assume, to the contrary, $\gamma(G) \leq n-1$, let D be a dominating set of G with $|D| = \gamma(G) \leq n-1$. Deleting some edges from G, we can get the spanning subgraph G_1 satisfying the following three properties:

(1) $\forall x \in V(G) - D, |N(x) \cap D| = 1$.

(2) $G_1[V(G) - D]$ has no any edges.

(3) $G_1[D]$ has no any cycles.

Obviously, G_1 is a forest and D is its dominating set. By adding some edges of G to G_1, we can get a spanning tree T of G, clearly, D is a dominating set of T, namely, $\gamma(T) \leq |D| \leq n-1$. By Theorem 3.1, $T \notin \tilde{T}_1 \cup \tilde{T}_2$, a contradiction. We have finished the proof of Theorem 3.2.

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具有最大控制数的连通图的刻画

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摘 要: 设 G 为一个 P 阶图， $\gamma(G)$ 表示 G 的控制数. 显然 $\gamma(G) \leq [\frac{p}{2}]$. 本文的目的是刻画达到这个上界的连通图. 主要结果： (1) 当 p 为偶数时， $\gamma(G) = \frac{p}{2}$ 当且仅当 $G \cong C_4$ 或者 G 为某连通图的冠； (2) 当 p 为奇数时， $\gamma(G) = \frac{p-1}{2}$ 当且仅当 G 的每棵生成树为定理 3.1 中所示的两类树之一.

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