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Global Stability in a Differential Equation with Piecewisely Constant Arguments *

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Abstract: In this paper we consider the differential equation with piecewisely constant arguments

$$N'(t) = r(t)(-\mu N(t) + \sum_{i=0}^{m} P_i e^{-r_i N([t-i])}), \quad t \geq 0,$$

where $[\cdot]$ denotes the greates integer function, $r(t) \in C([0,+\infty),(0,+\infty)), P_i \in [0,+\infty)(i = 1,2,\cdots,m)$, with $P_m > 0$, we establish some new sufficient conditions for an arbitrary solution $N(t)$ to satisfy the initial conditions of the form

$$N(0) = N_0 > 0 \text{ and } N(-j) = N_{-j} \geq 0, j = 1,2,\cdot,m,$$

to converge to the positive equilibrium N^* as $t \to \infty$.

Key words: differential equation; global stability; piecewise constant argument.

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1. Introduction

It is known that the delay differential equation

$$N'(t) = -\mu N(t) + pe^{-rN(t-\tau)}, \quad t \geq 0, \tag{1}$$

was used to model the survival of red blood cells[1].

Equation (1) has been extensively discussed, see [1-3]. Our aim in this paper is to consider the more general equation with piecewisely constant arguments

$$N'(t) = r(t)(-\mu N(t) + \sum_{i=0}^{m} P_i e^{-r_i N([t-i])}), \quad t \geq 0, \tag{2}$$

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where $[P]$ is the greatest integer $\leq P, r(t) \in C([0,+\infty),(0,+\infty)), r_j, P_j \geq 0 (j = 0,1,2,\cdots,m-1), P_m > 0, r_m > 0$, and $\mu > 0$.

By a solution of (2) we mean a function $N(t)$ which is defined on the set $\{-m,-m+1,\cdots,0\} \cup (0,+\infty)$, and possesses the following properties:

(i) N is continuous on $(0,+\infty)$;

(ii) The derivative $N'(t)$ exists at each point $t \in (0,+\infty)$ with the possible exception of the points $t \in \{0,1,2,\cdots\}$ where left-sided derivatives exist;

(iii) (2) is satisfied on each interval $(n, n+1)$ with $n = 0,1,2,\cdots$.

Using a method similar to that in Lemma 2.1 of [3], one can easily show that (2) together with initial conditions of the form

$$N(0) = N_0 > 0 \text{ and } N(-j) = N_{-j} \geq 0, j = 1,2,\cdots,m, \tag{3}$$

has a unique solution $N(t)$ which is positive for all $t \geq 0$, and (2) has a unique positive equilibrium N^* which satisfies $-\mu N^* + \sum_{i=0}^{m} P_i e^{-r_i N^*} = 0$.

2. Boundedness results

In this section, we consider conditions under which solutions of (2) will be bounded.

Lemma 1 *Let $N(t)$ be the solution of (2) and (3), if $N(t)$ is eventually greater (or less) than N^*, then the limit $\lim_{n\to\infty} N(t)$ exists and is positive. Furthermore if*

$$\int_0^\infty r(s)\mathrm{d}s = \infty, \tag{4}$$

then $\lim_{t\to\infty} N(t) = N^*$.

The proof is simple and that is omitted.

Lemma 2 *Assume that $N(t)$ is a solution of (2) and (3), and it is oscillatory about N^*, if for some $M > 0$, we have*

$$\limsup \int_{t-m}^{t} r(s)\mathrm{d}s = M. \tag{5}$$

Then $N(t)$ is bounded above and is bounded below away from 0.

Proof For any $M' > M$, there exists $T > m$ such that

$$\int_{t-m}^{t} r(s)\mathrm{d}s < M', \quad t > T,$$

We prove first that $N(t)$ is bounded above. Suppose that $\limsup N(t) = \infty$, since $N(t)$ is both unbounded and oscillatory, there exists a $t^* \geq m + T$, such that

$$N(t^*) = \max_{0\leq t\leq t^*} N(t) > N^*.$$

Let $D^- x(t)$ denote the left-side derivative of $x(t)$. If $t^* \notin \{0,1,\cdots\}$ then

$$D^- N(t^*) = r(t^*)(-\mu N(t^*) + \sum_{i=0}^{m} P_i e^{-r_i N([t-i])}) \geq 0,$$

we have again $\sum_{i=0}^{m} P_i e^{-r_i N^*} = \mu N^*$. Thus $N([t^*-i]) < N^*$. Then there exists $\xi \in [[t^*-m], t^*]$ such that $N(\xi) = N^*$ and $N(t) > N^*$ for $t \in (\xi, t^*]$. Integrating (2) from ξ to t^*, we have

$$\int_{\xi}^{t^*} [N'(t) e^{\int_0^t \mu r(s) ds} + \mu r(t) N(t) e^{\int_0^t \mu r(s) ds}] dt \le (\sum_{i=0}^{m} P_i) \int_{\xi}^{t^*} r(t) e^{\int_0^t \mu r(s) ds} dt,$$

$$N(t^*) \le N^* e^{\int_{t^*}^{\xi} \mu r(s) ds} + (\sum_{i=0}^{m} P_i) \int_{\xi}^{t^*} r(t) e^{\int_{t^*}^{t} \mu r(s) ds} dt \le N^* + (\sum_{i=0}^{m} P_i) \int_{[t^*-m]}^{t^*} \mu r(t) dt$$
$$\le N^* + (\sum_{i=0}^{m} P_i) \int_{[t^*]-m}^{[t^*]+1} \mu r(t) dt \le N^* + (\sum_{i=0}^{m} P_i) M' \mu.$$

If $t^* \in \{0, 1, 2, \cdots\}$, then

$$0 \le D^- N(t^*) = r(t^*)(-\mu N(t^*) + \sum_{i=0}^{m} P_i e^{-r_i N(t^*-i-1)}),$$

Thus there exists $\xi \in [t^*-m-1, t^*]$ such that $N(\xi) = N^*$ and $N(t) > N^*$ for $t \in (\xi, t^*]$. Similary, we have $N(t^*) \le N^* + (\sum_{i=0}^{m} P_i) M' \mu$.

Conseqently limsup$N(t) \le N^* + (\sum_{i=0}^{m} P_i) M \mu$. This contradiction shows that $N(t)$ is bounded above, and

$$N(t) \le N^* + \sum_{i=0}^{m} P_i M \mu. \tag{6}$$

Substituting (6) into (2), we have

$$N'(t) \ge r(t)(-\mu(N^* + \sum_{i=0}^{m} P_i M \mu) + \sum_{i=0}^{m} P_i e^{-r_i (N^* + \sum_{i=0}^{m} P_i M \mu)}), \quad t \ge 2m.$$

Next we will show that $N(t)$ is bounded below away from 0. Suppose that $\liminf N(t) = 0$, since $N(t)$ is oscillation about N^*, there exists $t^* > 3m$ such that $N(t^*) = \min_{0 \le t \le t_*} N(t) < N^*$ clearly $D^- N(t_*) \le 0$. Further more if $t_* \notin \{0, 1, 2, \cdots\}$, then $D^- N(t_*) = r(t_*)(-\mu N(t_*) + \sum_{i=0}^{m} P_i e^{-r_i N([t_*-i])} < 0$. Thus $\sum_{i=0}^{m} P_i e^{-r_i N([t_*-i])} \le \mu N(t_*) < \mu N^*$, and there exists an $N([t_*-i]) > N^*$ and an $\eta \in ([t_*-m], t_*)$, such that $N(\eta) = N^*$ and $N(t) < N_*$ for $t \in (\eta, t_*]$. Integrating (2) from ξ to t_*, we have

$$N(t_*) = N^* e^{\int_{t_*}^{\eta} \mu r(s) ds} + \int_{\eta}^{t_*} r(t) \sum_{i=0}^{m} P_i e^{-r_i N([t-i])} e^{\int_{t_*}^{t} \mu d\mu} ds \ge N^* e^{-\int_{\eta}^{t^*} \mu r(s) ds} \ge N^* e^{-\mu M'}.$$

If $t_* \in \{0, 1, 2, \cdots\}$, similary we have $N(t_*) \ge N^* e^{-\mu M'}$. Thus liminf$N(t) \ge N^* e^{-\mu M} > 0$, this is a contradiction. Hence the proof is complete. □

3. Global stability results

In this section we provide sufficient conditions for the global stability of the pisitive equilibrium N^* of (2). The main result is

Theorem 1 *Assume that* $R = \max\{r_i\}, r = \min\{r_i\}$,

$$\limsup \int_{t-m}^{t} \mu r(s) = M, \tag{7}$$

$$(1 - e^{-M})\frac{1}{\mu}\sum_{i=0}^{m} P_i r_i e^{-r_i N^*} \le 1, \tag{8}$$

$$\frac{R}{r^2}(1 - e^{-M})\frac{1}{\mu}\sum_{i=0}^{m} P_i r_i^2 e^{-r_i N^*} \le 1, \tag{9}$$

$$\int_0^{\infty} r(s)\mathrm{d}s = \infty. \tag{10}$$

Then the solution $N(t)$ *of (2) and (3) satisfies*

$$\lim_{t\to\infty} N(t) = N^*. \tag{11}$$

Proof In view of Lemma 1, it suffices to prove that (11) holds for all solution $N(t)$ which is oscillation about N^*. By Lemma 2, $N(t)$ is bounded above and bounded below away from 0. Set

$$u = \limsup N(t), \quad v = \liminf N(t), \tag{12}$$

then $0 < v \le N^* \le u < \infty$. It suffices to prove that $u = v = N^*$, for and $\varepsilon \in (0, v)$, choose an integer $T = T(\varepsilon) > 0$ such that

$$v_1 = v - \varepsilon < N(t - m) < u + \varepsilon = u_1 \text{ for } t > T. \tag{13}$$

Using (2) we have

$$N'(t) \le r(t)(-\mu N(t) + \sum_{i=0}^{m} P_i e^{-r_i v_1}), \quad t \ge T, \tag{14}$$

$$N'(t) \ge r(t)(-\mu N(t) + \sum_{i=0}^{m} P_i e^{-r_i u_1}), \quad t \ge T. \tag{15}$$

Let $\{T_n\}$ be an increasing sequence such that $T_n > T + 2m, D^- N(T_n) \ge 0, N(T_n) > N^*, \lim_{n\to\infty} N(T_n) = u$ and $\lim_{n\to\infty} T_n = \infty$. If $T_n \notin [0, 1, 2, \cdots\}$, by (2) we have

$$\sum_{i=0}^{m} P_i e^{-r_i N([T_n - i])} \ge \mu N(T_n).$$

Thus there exists $\xi_n \in ([T_n - m], T_n)$ such that $N(\xi_n) = N$ and $N(t) > N^*$ for $t \in (\xi_n, T_n]$. If $T_n \in \{0, 1, 2, \cdots\}$. Then by (2) we know that there exists $\xi_n \in [T_n - m - 1, T_n)$ such that $N(\xi_n) = N^*$ and $N(t) > N^*$ for $t \in (\xi_n, T_n)$, thus by (7), we have $\mu \int_{\xi_n}^{T_n} r(s)\mathrm{d}s < M'$.

By integrating (12) from ξ_n to T_n, we get

$$\begin{aligned} N(T_n) &\le N(\xi_n)e^{\int_{T_n}^{\xi_n}\mu r(s)\mathrm{d}s} + \int_{\xi_n}^{T_n} r(s)\sum_{i=0}^{m} P_i e^{-r_i v_1} e^{\int_{T_n}^{s}\mu r(u)\mathrm{d}u}\mathrm{d}s \\ &\le N^* e^{\int_{T_n}^{\xi_n}\mu r(s)\mathrm{d}s} + \int_{\xi_n}^{T_n} r(s)\sum_{i=0}^{m} P_i e^{-r_i v_1} e^{\int_{T_n}^{s}\mu r(u)\mathrm{d}u}\mathrm{d}s \\ &= (N^* - \frac{1}{\mu}\sum_{i=0}^{m} P_i e^{-r_i v_1})e^{\int_{T_n}^{\xi_n}\mu r(s)\mathrm{d}s} + \frac{1}{\mu}\sum_{i=0}^{m} P_i e^{-r_i v_1}, \end{aligned}$$

since $v_1 \le N^*$, thus $N^* - \frac{1}{\mu}\sum_{i=0}^{m} P_i e^{-r_i v_1} \le N^* - \frac{1}{\mu}\sum_{i=0}^{m} P_i e^{-r_i N^*} = 0$, $N(T_n) \le (N^* - \frac{1}{\mu}\sum_{i=0}^{m} P_i e^{-r_i v_1})e^{-M'} + \frac{1}{\mu}\sum_{i=0}^{m} P_i e^{-r_i v_1}$, let $n \to \infty, \varepsilon \to 0, M' \to M$, we have

$$u \le (N^* - \frac{1}{\mu}\sum_{i=0}^{m} P_i e^{-r_i v})e^{-M} + \frac{1}{\mu}\sum_{0}^{m} P_i e^{-r_i v}. \tag{16}$$

Let $\{t_n\}$ be an increasing sequence, such that $t_n > T + 2m, D^- N(T_n) \le 0, N(t_n) < N^*, \lim N(t_n) = v$ and $\lim_{n\to\infty} t_n = \infty$. If $t_n \notin \{0, 1, 2, \cdots\}$, we have by (2) that $\sum_{i=0}^{m} P e^{-r_i N([t_n - i])} \le \mu N(t_n)$. Thus there exists $\eta_n \in [[t_n - m], t_n]$, such that $N(\eta_n) = N^*$ and $N(t) < N^*$ for $t \in [\eta_n, t_n)$. If $t_n \in \{0, 1, 2, \cdots\}$, then by (2) we know that there exists $\eta_n \in (t_n - m - 1, t_n)$, such that $N(\eta_n) = N^*$ and $N(t) < N^*$ for $t \in (\eta_n, t_n)$. Thus by integrating (14) from η_n to t_n, we have

$$\begin{aligned} N(t_n) &\ge N^* e^{\int_{t_n}^{\eta_n}\mu r(s)\mathrm{d}s} + \frac{1}{\mu}\int_{\eta_n}^{t_n}\sum_{i=0}^{m} P_i e^{-r_i u_1}\mathrm{d}e^{\int_{t_n}^{s}\mu r(u)\mathrm{d}u} \\ &\ge (N^* - \frac{1}{\mu}\sum_{i=0}^{m} P_i e^{-r_i u_1})e^{\int_{t_n}^{\eta_n}\mu r(u)\mathrm{d}u} + \frac{1}{\mu}\sum_{i=0}^{m} P_i e^{-r_i u_1}, \end{aligned}$$

since $u_1 > u \ge N^*$, thus $N^* - \frac{1}{\mu}\sum_{i=0}^{m} P_i e^{-r_i u_1} \ge N^* - \frac{1}{\mu}\sum_{i=0}^{m} P_i e^{-r_i N^*} = 0$. Then

$$N(t_n) \ge (N^* - \frac{1}{\mu}\sum_{i=0}^{m} P_i e^{-r_i u_1})e^{-M'} + \frac{1}{\mu}\sum_{i=0}^{m} P_i e^{-r_i u_1}.$$

Let $n \to \infty, \varepsilon \to 0, M' \to M$,

$$v \ge (N^* - \frac{1}{\mu}\sum_{i=0}^{m} P_i e^{-r_i u})e^{-M} + \frac{1}{\mu}\sum_{i=0}^{m} P_i e^{-r_i u}. \tag{17}$$

From (16), we have $u - N^* e^{-M} + (1 - e^{-M})\frac{1}{\mu}\sum_{i=0}^{m} P_i e^{-r_i v} \le 0$.

Set $f(v) = u - N^* e^{-M} + (1 - e^{-M})\frac{1}{\mu}\sum_{i=0}^{m} P_i e^{-r_i v}$, from (16) and (17), we have

$$f'(v) = -\frac{1}{(1 - e^{-M})\frac{1}{\mu}\sum_{i=0}^{m} P_i r e^{-r_i u}} + (1 - e^{-M})\frac{1}{\mu}\sum_{i=0}^{m} P_i r_i e^{-r_i v}, \quad 0 < v < N^*,$$

$f(N^*)=0$. Now we prove that $f'(v)<0$. Set

$$g(v)=-1+(1-e^{-M})^2\frac{1}{\mu^2}(\sum_{i=0}^{m}P_ir_ie^{-r_iv})(\sum_{i=0}^{m}P_ir_ie^{-r_iu})$$

$$g(N^*)=-1+(1-e^{-M})^2\frac{1}{\mu^2}(\sum_{i=0}^{m}P_ir_ie^{-r_iN^*})^2\le 0 \quad (\text{by } (8)),$$

$$\begin{aligned} g'(v)=(1-e^{-M})^2\frac{1}{\mu^2}[-(\sum_{i=0}^{m}P_ir_i^2e^{-r_iv})(\sum_{i=0}^{m}P_ir_ie^{-r_iu})+ \\ \sum_{i=0}^{m}P_ir_ie^{-r_iv}\frac{\sum_{i=0}^{m}P_ir_i^2e^{-r_iu}}{(1-e^{-M})\frac{1}{\mu}\sum_{i=0}^{m}P_ir_ie^{-r_iu}}] \\ >(1-e^{-M})^2\frac{1}{\mu^2}[-\sum_{i=0}^{m}P_ir_i^2e^{-r_iN^*}\sum_{i=0}^{m}P_ir_ie^{-r_iN^*}+\frac{r^2\sum_{i=0}^{m}P_ir_ie^{-r_iN^*}}{R(1-e^{-M})\frac{1}{\mu}}]\ge 0 \quad (\text{by } (9)). \end{aligned}$$

If $v<N^*$, we have $g(v)<g(N^*)\le 0$, thus $f'(v)<0$ and $f(v)>f(N')=0, u>N^*e^{-M}+\frac{1}{\mu}(1-e^{-M})\sum_{i=0}^{m}P_ie^{-r_iv}$.

This contradicts to (16). Thus $v=N^*$. Similary, we have $u=N^*$. The proof is completed. □

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具分片常变量泛函微分方程的全局稳定性

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摘 要: 本文研究具分片常变量泛函微分方程 $N'(t)=r(t)(-\mu N(t)+\sum_{i=0}^{m}P_ie^{-r_iN([t-i])})$, $t\ge 0$, 其中 $[\cdot]$ 表示取整函数, $r(t)\in C([0,+\infty),(0,+\infty))$, $P_i\in[0,+\infty)$, $(i=1,2,\cdots,m)$, $P_m>0$, 文中给出了保证方程的每一满足初始条件 $N(0)=N_0>0, N(-j)=N_{-j}\ge 0$ $(j=1,2,\cdots,m)$, 的解 $N(t)$ 满足 $\lim_{t\to\infty}N(t)=N^*$ 的一些新的充分条件.