Journal of Mathematical Research & Exposition
Vol.21, No.1, 47–52, February, 2001

Bifurcation of A Class of Reaction-Diffusion Equations *

LI Chang-pin
(Dept. of Math., Shanghai University, Shanghai 200436, China)

Abstract: The paper deals with one important class of reaction-diffusion equations, $u'' + \mu(u - u^k) = 0 (2 \leq k \in Z^+)$ with boundary value condition $u(0) = u(\pi) = 0$. Singularity theory based on the method of L-S (Liapunov-Schmidt) is applied to its bifurcation analysis. And the satisfactory results are obtained.

Key words: reaction-diffusion equation; L-S reduction; singularity method; bifurcation.

Classification: AMS(1991) 34B15,34C23/CLC O175.12,O175.14,O177.91

Document code: A **Article ID:** 1000-341X(2001)01-0047-06

1. Introduction

One class of reaction-diffusion equation ([2], [7]) is read as,

$$F(u, \mu) = u'' + \mu(u - u^k) = 0 \tag{1.1}$$

with boundary value condition

$$u(0) = u(\pi) = 0, \tag{1.2}$$

where μ is a parameter and $2 \leq k \in Z^+$.

Singularity theory ([3], [5], [6]) plays an important role in static bifurcation analysis of nonlinear problems with some parameters. By using this theory, we obtain satisfactory results of (1.1). In details, we divide the paper into two sections apart from this introduction. In section 2, we apply L-S reduction ([1], [3], [5], [6], etc) to (1.1) at the bifurcation point $(u, \mu) = (0, n^2)$ to get bifurcation equation. Section 3 is devoted to the bifurcation analysis of the bifurcation equation obtained in section 2 for the cases $k = 2, 3$. With the increase of the exponent k of u, it is more and more difficult to deal with its bifurcation. Many efforts were made to study the bifurcated phenomenon of (1.1) and (1.2) for $k \geq 4$ in [4].

---

*Received date: 1998-10-19
Foundation item: Supported by the National Natural Science Foundation (19971057) and by the Youth Science Foundation of Shanghai Municipal Commission of Education (99QA66)
Biography: LI Chang-pin (1968-), male, Ph.D., Associate Professor.
E-mail: Leecp@online.sh.cn

2. L-S Reduction

Let $X = \{u|u \in C^2[0,\pi], u(0) = u(\pi) = 0\}$, $Y = C^0[0,\pi]$. We define inner products on these spaces by $\langle u, v\rangle = \int_0^\pi u(\xi)v(\xi)\mathrm{d}\xi$. Then $F(u,\mu)$ is a map from $X \times R$ onto Y. For every μ, (1.1) has a trivial solution $u = 0$, i.e., $F(0,\mu) \equiv 0$.

Consider the linearized equation of (1.1)

$$D_uF(0,\mu)v = v'' + \mu v = 0 \tag{2.1}$$

with boundary value condition

$$v(0) = v(\pi) = 0. \tag{2.2}$$

It is easy to show that (2.1) and (2.2) has nontrivial solutions $v = c\sin nx$(c is an arbitrary constant) iff $\mu = \mu_n = n^2 (n \in Z^+)$, and that (2.1) and (2.2) has trivial solution if $\mu \neq \mu_n$.

Let $L_n = (dF)_{(0,n^2)} = \frac{d^2}{\mathrm{d}x^2} + n^2$, ker $L_n = \mathrm{span}\{\sin n\xi\} = \mathrm{span}\{e\}$. It is known to us that $L_n : X \to Y$ is a Fredholm operator of index zero ([3]). Also, we know that L_n is a self-adjoint operator, i.e. $L_n^* = L_n$. Now we can split the domain of F into

$$X = \ker L_n \oplus M, \quad Y = N \oplus \mathrm{range} L_n,$$

where $M = (\ker L_n)^\perp$, $N = (\mathrm{range} L_n)^\perp (= \ker L_n^* = \ker L_n)$.

Let P_e be the orthogonal projector from Y onto rangeL_n, which is $P_e u = u - \langle u, e\rangle e, u \in Y$. By L-S reduction, (1.1) is equivalent to

$$P_eF(v + w, \mu) = 0, \quad v \in \ker L_n, \ w \in M \tag{2.3}$$

$$(I - P_e)F(v + w, \mu) = 0. \tag{2.4}$$

Equation (2.3) is solved for a unique $w(v,\mu)(w(0,n^2) = 0)$ by *implicit function theorem* (for instance, see [1]). Substituing $w(v,\mu)$ into (2.3) yields

$$(I - P_e)F(v + w(v,\mu),\mu) = 0, \tag{2.5}$$

which is called the bifurcation equation.

Taking the inner product of (2.5) with e and letting $v = xe$, we have

$$\langle e, P_eF(xe + w(x,\mu),\mu)\rangle = 0. \tag{2.6}$$

(2.6) is also called the bifurcation equation of (1.1) at $(u,\mu) = (0,n^2)$. Since we cannot find the expression of $w(x,\mu)$, hence, it is impossible to find the solution to (2.6). In next section, we deal with its bifurcations by using the singularity theory.

3. Bifurcation Analysis

It is necessary to introduce some useful definitions and some important results([3], [5], [6]) for our later discussion.

Let $E_{x,\mu} = \{g|g :\ C^\infty$ map from $R^2 \times R$ onto R on some neighborhood of $(0,0)\}$. We shall identify any two functions in $E_{x,\mu}$ which are equal on some neighborhood of (0, 0).

We call the elements of $E_{x,\mu}$ germs. In fact , a germ is an equivalence class in $E_{x,\mu}$ with respect to this identification. Obviously, $E_{x,\mu}$ is a linear space.

Suppose

$$f(x,\mu)=0, \quad (x,\mu)\in U\times V\subset R^2, \tag{3.1}$$

where $f\in E_{x,\mu}$, $(0,0)\in U\times V$ and $f(0,0)=f_x(0,0)=0$, i.e., (0, 0) is a singular point of f.

For $f,h\in E_{x,\mu}$, we shall say that f and h are equivalence if

$$f(x,\mu)=S(x,\mu)h(\Omega(x,\mu)), \tag{3.2}$$

where $\Omega(x,\mu)=(X(x,\mu),\Lambda(\mu))$ is a C^∞ diffemorphism on the neighborhood of the origin, $S(x,\mu)\in E_{x,\mu}$ and $X(0,0)=\Lambda(0)=0$, $\Lambda'(0)>0$, $X_x(0,0)>0$, $S(0,0)>0$. If $\Lambda(\mu)\equiv\mu$, then we say that f and h are strongly equivalence. The fact that f and h are equivalence implies that

(1) f and h have same singular points.

(2) $n_f(\mu)=n_h(\Lambda(\mu))$ ($n_f(\mu)$ denotes the number of solutions of $f(x,\mu)=0$).

(3) The stability of the equilibrium solution of $\dot{x}=g(x,\mu)$ is the same as that of $\dot{x}=h(x,\mu)$.(The reason is $X_x>0$, $S>0$ at some neighborhood of the origin.)

The following theorem is important (See, for instance, [2], etc).

Theorem 3.1 *A germ $f\in E_{x,\mu}$ is strongly equivalent to $\varepsilon x^k+\delta\mu x$ iff at $x=\mu=0$,*

$$f=f_x=\cdots=f_{x^{k-1}}=f_\mu=0 \tag{3.3}$$

and

$$\varepsilon=\mathrm{sgn} f_{x^k},\ \delta=\mathrm{sgn} f_{x\mu}. \tag{3.4}$$

For small k, the bifurcation of $\dot{x}=\varepsilon x^k+\delta\mu x$ at $(x,\mu)=(0,0)$ is known to us. For example,

(I) $(x,\mu)=(0,0)$ is a transcritical bifurcation point of $\dot{x}=-x^2+\mu x$, its bifurcation diagram is Fig.3.1.

(II) $(x,\mu)=(0,0)$ is a pitchfork bifurcation point of $\dot{x}=-x^3+\mu x$, its bifurcation diagram is Fig.3.2.

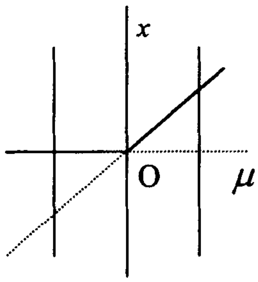

Fig.3.1 Transcritical bifurcation diagram

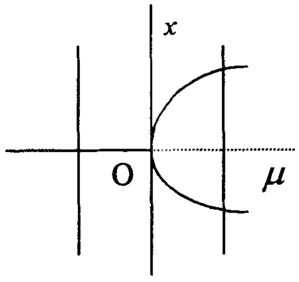

Fig.3.2 Pitchfork bifurcation diagram

Assuming $v_i \in R^n (i = 1, 2, \cdots, k)$, we define

$$(d^k G)_{(y,\alpha)}(v_1, \cdots, v_k) = \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k} G(y + \sum_{i=1}^{k} t_i v_i, \alpha)|_{t_1 = \cdots = t_k = 0}. \tag{3.5}$$

Obviously, $(d^k G)_{(y,\alpha)}$ is a symmetric, multilinear function of k arguments.
Now let us compute the partial derivatives of $g(x, \mu)$ defined by (2.6).

$$g_x = \langle e, dF(e + w_x) \rangle, \tag{3.6}$$

$$g_{x^2} = \langle e, dF(w_{x^2}) + d^2F(e + w_x, e + w_x) \rangle, \tag{3.7}$$

$$g_{x^3} = \langle e, dF(w_{x^3}) + 3d^2F(e + w_x, w_{x^2}) + d^3F(e + w_x, e + w_x, e + w_x) \rangle, \tag{3.8}$$

$$\cdots$$

$$g_\mu = \langle e, dF(w_\mu) + F_\mu \rangle, \tag{3.9}$$

$$g_{x\mu} = \langle e, dF_\mu(e + w_x) + dF(w_{x\mu}) + d^2F(e + w_x, w_\mu) \rangle, \tag{3.10}$$

$$\cdots.$$

We rewrite (2.3) as

$$P_e F(xe + w(x, \mu), \mu) = 0. \tag{3.11}$$

Differentiating (3.11) with respect to x and μ leads to

$$P_e dF(e + w_x) = 0, \tag{3.12}$$

$$P_e d^2F(e + w_x, e + w_x) + P_e dF(w_{x^2}) = 0, \tag{3.13}$$

$$P_e d^3F(e + w_x, e + w_x, e + w_x) + 3P_e d^2F(e + w_x, w_{x^2}) + P_e dF(w_{x^3}) = 0, \tag{3.14}$$

$$\cdots$$

$$P_e dF(w_\mu) + P_e F_\mu = 0, \tag{3.15}$$

$$P_e dF_\mu(e + w_x) + P_e dF(w_{x\mu}) + P_e d^2F(e + w_x, w_\mu) = 0, \tag{3.16}$$

$$\cdots.$$

The partial derivatives of $F(u, \mu)$ (given by (1.1)) with respect to u and μ at $(0, n^2)$ are obtained,

$$\begin{aligned}(d^2F)_{(0,n^2)}(\xi_1, \xi_2) &= \frac{\partial^2}{\partial t_1 \partial t_2}[t_1 \xi_1'' + t_2 \xi_2'' + n^2(t_1\xi_1 + t_2\xi_2 - (t_1\xi_1 + t_2\xi_2)^k)]|_{t_1 = t_2 = 0} \\ &= \begin{cases} -2n^2 \xi_1 \xi_2 & k = 2 \\ 0 & k > 2, \end{cases}\end{aligned} \tag{3.17}$$

$$(d^3F)_{(0,n^2)}(\xi_1, \xi_2, \xi_3) = \begin{cases} 0 & k = 2, \\ -6n^2 \xi_1 \xi_2 \xi_3 & k = 3, \\ 0 & k > 3. \end{cases} \tag{3.18}$$

Let (3.12) evaluate on $(0,n^2)$, then $P_eL_n(e+w_x(0,n^2)) = L_nw_x(0,n^2) = 0$. It follows that

$$w_x(0,n^2) = 0. \quad (L_n : M \to \text{range}L_n \text{ is invertible.}) \tag{3.19}$$

Using (3.6) and (3.19), then

$$g_x(0,n^2) = 0. \tag{3.20}$$

Applying (3.13) evaluated on $(0,n^2)$ and substituting (3.19) into (3.13) leads to

$$w_{x^2}(0,n^2) = \begin{cases} -L_n^{-1}(-2n^2)P_e e^2 & k=2 \\ 0 & k>2. \end{cases} \tag{3.21}$$

Let (3.7) evaluate on $(0,n^2)$ and substitute (3.17), (3.19), (3.25) into (3.7), then

$$\begin{aligned} g_{x^2}(0,n^2) &= \begin{cases} \langle e, L_n(2n^2)L_n^{-1}P_e e^2 - 2n^2e^2\rangle & k=2 \\ 0 & k>2 \end{cases} \\ &= \begin{cases} \dfrac{4n[(-1)^n-1]}{3} & k=2 \\ 0 & k>2. \end{cases} \end{aligned} \tag{3.22}$$

From (3.14), (3.18), (3.19) and (3.21), we can obtain

$$w_{x^3}(0,n^2) = \begin{cases} 12n^4L_n^{-1}(P_e(eL_n^{-1}P_e e^2)) & k=2 \\ 6n^2L_n^{-1}(P_e e^3) & k=3 \\ 0 & k>3. \end{cases} \tag{3.23}$$

Let $(u,\mu) = (0,n^2)$, apply (3.17) and (3.18), and take (3.19), (3.21), (3.23) into (3.8), we can get

$$\begin{aligned} g_{x^3}(0,n^2) &= \begin{cases} \langle e, -12n^4e(L_n^{-1}P_e e^2)\rangle & k=2 \\ \langle e, -6n^2e^3\rangle & k=3 \\ 0 & k>3 \end{cases} \\ &= \begin{cases} \begin{cases} -\dfrac{5\pi n^2}{2} & n \text{ is even} \\ \text{omitted} & n \text{ is odd} \end{cases} & k=2 \\ -\dfrac{9\pi n^2}{4} & k=3 \\ 0 & k>3. \end{cases} \end{aligned} \tag{3.24}$$

From (3.15), we can get $P_eL_nw_\mu(0,n^2) = 0$, i.e.,

$$w_\mu(0,n^2) = 0. \tag{3.25}$$

Also from (3.16), we can get $w_{x\mu}(0,n^2) = 0$. It immediately follows that

$$g_\mu(0,n^2) = 0, g_{x\mu}(0,n^2) = \langle e, e\rangle = \frac{\pi}{2}.$$

Now, let us summarize our results.

Case 1 $k=2(n\in Z^+)$

1^0 *n is odd. At* $(u,\mu)=(0,n^2)$, $g=g_x=g_\mu=0$, $g_{x^2}=-\dfrac{8n}{3}$, $g_{x\mu}=\dfrac{\pi}{2}$.

2^0 *n is even.*

At $(u,\mu)=(0,n^2)$, $g=g_x=g_{x^2}=g_\mu=0$, $g_{x^3}=-\dfrac{5\pi n^2}{2}$, $g_{x\mu}=\dfrac{\pi}{2}$.

According to theorem 3.1, we have

Theorem 3.2 *Suppose* $k=2$ *and* $n\in Z^+$. *If n is odd, then* $g(x,\mu)$ *is strongly equivalent to* $-x^2+(\mu-n^2)x$. *Furthermore,* $(u,\mu)=(0,n^2)$ *is a transcritical bifurcation point of (1.1). The bifurcation diagram of* $\dfrac{\partial u}{\partial t}=F(u,\mu)$ *at* $(0,n^2)$ *is similar to Fig.3.1. If n is even, then* $g(x,\mu)$ *is strongly equivalent to* $-x^3+(\mu-n^2)x$. *Furthermore,* $(u,\mu)=(0,n^2)$ *is a pitchfork point of (1.1). The bifurcation diagram of* $\dfrac{\partial u}{\partial t}=F(u,\mu)$ *at* $(0,n^2)$ *is similar to Fig.3.2.*

Case 2 $k=3(n\in Z^+)$

At $(u,\mu)=(0,n^2)$, $g=g_x=g_{x^2}=g_\mu=0$, $g_{x^3}=-\dfrac{9\pi n^2}{4}$, $g_{x\mu}=\dfrac{\pi}{2}$.

Theorem 3.3 *Suppose* $k=3$ *and* $n\in Z^+$. $g(x,\mu)$ *is strongly equivalent to* $-x^3+(\mu-n^2)x$. *Furthermore,* $(u,\mu)=(0,n^2)$ *is a pitchfork point of (1.1). The bifurcation diagram of* $\dfrac{\partial u}{\partial t}=F(u,\mu)$ *at* $(0,n^2)$ *is similar to Fig.3.2.*

Till now, in this paper and [4], we have finished analyzing the bifurcations of (1.1) and (1.2) for all positive integers except $k=1$ which belongs to the trivial case.

References:

[1] CHOW S N & HALE J K. *Methods of Bifurcation Theory* [M]. Springer-Verlag, 1982.

[2] FIFE P C. *Mathematical Aspects of Reaction and Duffusing Systems* [M]. Lecture Notes in Biomathematics, 1979, **28**.

[3] GOLUBITSKY M & SCHAEFFER D G. *Singularities and Groups in Bifurcation Theory* [M]. Springer-Verlag, 1985, **I**.

[4] LI Chang-pin. *A Note of Bifurcations of* $u''+\mu(u-u^k)=0$ $(4\leq k\in Z^+)$ [J]. Appl. Math. Mech., 2000, **21**: 265–274.

[5] LU Q S. *Bifurcation and Singularity* [M]. Shanghai: Shanghai Scientific and Technological Education Publishing House, 1995.

[6] TANG Y. *Foundations of Symmetry Bifurcation Theory* [M]. Chinese Science Press, 1998.

[7] YE Q X& LI Z Y. *Introduction to Reaction-diffusion Equations* [M]. Chinese Science Press, 1994.

一类反应扩散方程的分歧

李 常 品

(上海大学数学系，上海 200436)

摘 要: 本文讨论了一类反应扩散方程的分歧现象. 运用所谓基于李雅普诺夫施密特约化的奇异理论方法，得到了满意的结果.