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An Inverse Problem for a Nonlinear Evolution Equation *

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Abstract: This paper deals with an inverse problem for the unknown source term in a nonlinear evolution equation. Firstly, the authors change the initial boundary value problem (IBVP) for the equation into a Cauchy problem for a certain nonlinear evolution equation. Secondly, using the semigroup theory, the authors establish the existence and uniqueness of the solution for the inverse problem. Finally, they take advantage of the fixed point method for some contraction mapping and get the solvability of the inverse problem for the evolution equation.

Key words: evolution equation; inverse problem; semigroup theory; fixed point method.

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1. Introduction

In this paper, we want to discuss the following inverse problem of a multidimensional nonlinear evolution equation

$$Lu_t = b\Delta u + f(x,t,u) + \varphi(t)p(x,t), \qquad (x,t) \in \Omega \times (0,T], \tag{1.1}$$

where

$$Lu = (I - \Delta)u, \tag{1.2}$$

with the following initial value and boundary value conditions:

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{1.3}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{1.4}$$

$$u(x_0,t) = r(t), \qquad x_0 \in \Omega, 0 < t < T, \tag{1.5}$$

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where the bounded domain $\Omega \subset R^n = \overbrace{R^1 \times R^1 \times \cdots R^1}^{n}$ and $R^1 = (-\infty, +\infty)$. $\partial\Omega$ is the smooth boundary of Ω. I is the identity transformation, Δ is the n-dimensional Laplace operator. The diffusion coefficient $b > 0$ is a constant or a bounded continuous function on $\Omega \times R^+$, $f(x,t,p)$ is a nonlinear smooth function which has been defined on $\Omega \times R^+ \times R^1, R^+ = [0, +\infty)$; $\varphi(t), u_0(x)$ and $r(t)$ are given functions,but $u(x,t)$ and $p(x,t)$ are unknown functions.

The main aim of this paper is to look for the function pair $\{u(x,t), p(x,t)\}$ to satisfy the inverse problem (1.1)–(1.5).For this purpose, we shall make use of semigroup theory and fixed point method which are different from the arguments in documents [1–3].

To simplify the discussion, we utilize the same symbol $H^k(\Omega)$ to stand for the usual Sobolev space [4, 5].Specially, we say $H^0(\Omega) = L^2(\Omega)$ and $H_0^m(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$ (also c.f. [4] or [5]).

Besides, we introduce the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|_k$ as follows

$$(u,v)_0 = \int_\Omega uv\mathrm{d}x, \|u\|_k = (\sum_{\|\alpha\|\le k} \|D^\alpha u\|_0^2)^{\frac{1}{2}}, \quad \|u\|_{L^\infty(\Omega)} = \text{ess} \sup_{x\in\Omega} |u|. \tag{1.6}$$

Throughout of the discusion of this paper, we suppose that for any $\xi = (\xi_1, \cdots, \xi_n) \in R^n, \xi \neq 0$, there exists a positive constant μ, such that

$$\sum_{i,j=1}^{n} b\xi_i\xi_j \ge \mu|\xi|^2, \qquad \forall(x,t) \in \overline{\Omega} \times R^+. \tag{1.7}$$

This paper is organized as follows. In section 2, we discuss the Cauchy problem of the equivalent equation of evolution to the direct problem (1.1)–(1.4). We make use of the semigroup theory and establish the existence and uniqueness of solutions of the Cauchy problem. In section 3, we state and prove the main result of this paper.

2. The equivalent cauchy problem and some lemmas

In the following statements, we always assume that $u(\cdot,t) = u(t); p(\cdot,t) = p(t); b(\cdot,t) = b(t) = b; f(\cdot,t,u(\cdot)) = f(t,u(t))$. It is easy to know $L = I - \Delta$ is a bounded linear operator for $t \ge 0$. So we can infer that there is the inverse operator L^{-1}. We denote $A = L^{-1}b(t)\Delta$. In this way, we can rewrite (1.1)–(1.4) as the following equivalent Cauchy problem of the evolution equation

$$\frac{\mathrm{d}\,u(t)}{\mathrm{d}\,t} + Au(t) = L^{-1}f(t,u(t)) + L^{-1}\varphi(t)p(t), \tag{2.1}$$

$$u(0) = u_0. \tag{2.2}$$

In order to solve problem (2.1) and (2.2), we consider a more general abstract Cauchy problem

$$\frac{d\,u(t)}{\mathrm{d}\,t} + Au(t) = f(t), \qquad \forall t > 0; \text{ with } u(0) = u_0. \tag{2.3}$$

It is known that there are many perfect results about Cauchy problem for linear evolution equation. We refer to [4,6] and other documents. Here, we quote one lemma of [6].

Lemma 1[6] *Assume that H is a Hilbert space, then $-A$ is the infinitesimal generator of some C_0-contractive semigroup $S(t)$ on H, if and only if A not only is a dense closed operator, but also is an accretive operator in H, besides, there exists a $\lambda > 0$, such that $\lambda I + A$ is a covering mapping.*

We shall prove the following important lemma.

Lemma 2 *Suppose $u \in H_0^1(\Omega), f(t) \in C([0,t]; H_0^1(\Omega)), L = I - \Delta, A = L^{-1}b(t)\Delta$. Then there is the following conclusion*

(i) Cauchy problem (2.3) has a unique solution $u(t) \in C^1([0,T]; H_0^1(\Omega)) \cap C([0,T]; H_0^1(\Omega) \cap H^2(\Omega))$ and $u(t) = S(t)u_0 + \int_0^t S(t-\tau)f(\tau)\,\mathrm{d}\tau$, where $S(t)$ is a C_0-contractive semigroup with its infinitesimal generator $-A$.

(ii) The above semigroup $S(t)$ is an analytic semigroup in a certain fan region Δ_θ.

Proof (i) Denote $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$. According to [6], $D(A)$ is dense in $H_0^1(\Omega)$. For any $g \in H^0(\Omega) = L^2(\Omega)$, we consider

$$(\lambda L + b(t)\Delta)u = g. \tag{2.4}$$

By the direct computing method, we can see, for all $\lambda > 0$, the inner product $(\lambda Lu + b(t)\Delta u, v)_0$ can be extended onto $H_0^1(\Omega) \times H_0^1(\Omega)$, such that this inner product has the following continuous bilinear form

$$B(u,v) = \lambda \int_\Omega (uv + \nabla u \nabla v)\mathrm{d}x + \int_\Omega b(t)\nabla u \nabla v \mathrm{d}x, \qquad \forall u, v \in H, \tag{2.5}$$

where (∇u) satisfies $\nabla(\nabla u) = \Delta u$ with ∇ is the n-dimensional gradient operator. On the basis of the assumption of $b(t)$, we know $B(u,v)$ is a forced bilinear[4], i.e., for any $u \in H_0^1(\Omega)$, there exists a constant $K > 0$, such that

$$B(u,u) \geq K\|u\|_1^2. \tag{2.6}$$

So from Lax-Milgram lemma and the smooth behaviour of $\partial\Omega$, we infer that equation(2.4) has a unique solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$. Set $g = L\varphi$, we apply L^{-1} to both sides of (2.4), then we get

$$(\lambda I + A)u = \varphi, \qquad \text{with } A = L^{-1}b(t)\Delta. \tag{2.7}$$

It is easy to see, for any $\varphi \in H_0^1(\Omega)$, (2.7) has a unique solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$. That is to say $\lambda I + A$ is a covering mapping.

For any $u \in D(A)$, any $v \in H_0^1(\Omega)$, from $A = \Delta A + b\Delta$ and $(-\Delta u, v)_0 = (\nabla u, \nabla v)_0$, we get $(Au,u)_1 = (Au,u)_0 + (\nabla Au, \nabla u)_0 = (\Delta Au, u)_0 + (b\Delta u, u)_0 + (\nabla Au, \nabla u)_0$. By the given condition, we deduce that

$$(Au,u)_1 = b(\Delta u, u)_0 \geq 0. \tag{2.8}$$

Therefore, A is an accretive operator [6].

According to Lemma 1 and the basic theorem (c.f.[6]), we can deduce $-A$ is the infinitesimal generator of the C_0-contractive semigroup $S(t)$.On this reason, Cauchy problem (2.3) has a unique solution $u \in C([0,T]; H_0^1(\Omega)) \cap C([0,T]; H_0^1(\Omega) \cap H^2(\Omega))$ and $u(t)$ can be expressed as $u(t) = S(t)u_0 + \int_0^t S(t-\tau)f(\tau)\mathrm{d}\tau$.

(ii) On the basis of the proof of (i) and Theorem 5.2 of Chapter 2 of [6], we need only to prove that there exists a $\theta \in (0, \frac{\pi}{2})$ and a constant $K^* > 0$,such that

$$\rho(-A) \supset \Sigma \overset{\text{def}}{=} \{\lambda \mid |\arg \lambda| < \frac{\pi}{2} + \theta\} \cup \{0\},$$

where $\rho(-A)$ is the predicative set of $-A$ with its preradical $R(\lambda, -A)$ satisfies the following relation $\|R(\lambda, -A)\| \leq \frac{K^*}{|\lambda|}, \qquad \forall \lambda \in \Sigma, \qquad \lambda \neq 0.$

In fact, if $\lambda = \sigma + i\tau, \sigma \geq 0, i = \sqrt{-1}$, then for any $u, v \in H_0^1(\Omega)$,

$$Re((Au,u)_1 + \lambda(u,u)_1) = (b\Delta u, u)_0 + \sigma(u,u)_1 \geq K\|u\|_1^2. \tag{2.9}$$

where $K > 0$ is a constant. Note the relation (2.8), we see

$$|(Au,v)_1 + \lambda(u,v)_1| = |(b\Delta u, v)_0 + \lambda(u,v)_1| \leq |(b\Delta u, v)_0| + |\lambda|\|u\|_1\|v\|_1, \tag{2.10}$$

$$|(b\Delta u, v)_0| \leq \int_\Omega |b||\nabla u||\nabla v|\mathrm{d}x. \tag{2.11}$$

By the estimated method of the basic inequality and (2.11), we can deduce that there exists a positive constant K_1, satisfies

$$|(b\Delta u, v)_0| \leq K_1\|u\|_1\|v\|_1. \tag{2.12}$$

So from (2.10) and (2.12), there is a constant $K_2 > 0$, such that

$$|(Au,v)_1 + \lambda(u,v)_1| \leq K_2\|u\|_1\|v\|_1. \tag{2.13}$$

By (2.9), we see the differential operator $\lambda I + A : H_0^1(\Omega) \to H_0^1(\Omega)$ is a covering mapping for $\lambda = \sigma + i\tau, \sigma \geq 0$.

If $u \neq 0, u \in H_0^1(\Omega)$, then $(\lambda I + A)u \neq 0$. That is to say, $\lambda I + A$ is a one to one mapping.

From (2.12) and (2.13), we may know $\lambda I + A$ is a bounded operator. So we get the existence and boundedness of $(\lambda I + A)^{-1}$ on the basis of the inverse theorem of bounded linear operator.

Specially, if $\lambda = 0$, then $0 \in \rho(-A)$, and by (2.9), we can also get $\|(\lambda I + A)u\|_1\|u\|_1 \geq \sigma\|u\|_1^2$. It follows that

$$\|\sigma(\lambda I + A)^{-1}\| \leq 1. \tag{2.14}$$

On the other hand, there is $\|\mathrm{Im}((\lambda I + A)u, v)_1\| = |\tau|\|u\|_1^2$. So we get

$$\|((\lambda I + A)u, u)_1\| = |\tau|\|u\|_1^2, \qquad \text{i.e.,}\ \|(\lambda I + A)u\|_1 = |\tau|\|u\|_1.$$

It shows

$$\|\tau(\lambda I+A)^{-1}\| \le 1, \qquad \text{for } \lambda=\sigma+i\tau, \sigma>0. \tag{2.15}$$

In (2.15), let $\lambda=\alpha=i\tau$, then $\|(\alpha I+A)^{-1}\| \le \frac{1}{|\alpha|}$. Hence, we are easy to see for $|\sigma|/|\tau|<1$,

$$\|(\lambda-\alpha)(\alpha I+A)^{-1}\| = \|\sigma(\alpha I+A)^{-1}\| \le \frac{|\sigma|}{|\tau|} < 1,$$

$$\|[I-(\lambda-\alpha)(\alpha I+A)^{-1}]^{-1}\| \le \Big(\frac{1-|\lambda-\alpha|}{|\alpha|}\Big)^{-1}.$$

The above relation is equivalent to

$$\begin{aligned}\|(\lambda I+A)^{-1}\| &= \|(\alpha I+A)^{-1}[I-(\alpha-\lambda)(\alpha I+A)^{-1}]^{-1}\| \\ &\le \frac{1}{|\alpha|}\Big[1-\frac{|\lambda-\alpha|}{|\alpha|}\Big]^{-1} = \frac{1}{|\tau|}\Big[1-\frac{|\sigma|}{|\tau|}\Big]^{-1}.\end{aligned} \tag{2.16}$$

In particular, let $\theta = arctg1 = \frac{\pi}{4}$, then we see $(\lambda I+A)^{-1}$ is exists for $|\sigma|/|\tau| = tg\theta = 1$. Because $0\in\rho(-A)$, then we have

$$\rho(-A)\supset\{\lambda \mid |arg\lambda|<\frac{\pi}{2}+\theta\}\cup\{0\}. \tag{2.17}$$

In addition, by (2.16) and (2.17), for $\lambda\ne 0$ and $\lambda\in\{\lambda \mid |arg\lambda|<\frac{3\pi}{4}\}$, there is the following inequality $\|\lambda(\lambda I+A)^{-1}\| \le \frac{|\sigma|+|\tau|}{|\tau|}(1-|\sigma|/|\tau|)^{-1} \le 2(1-|\sigma|/|\tau|)^{-1}$. Denote $K^*=2(1-|\sigma|/|\tau|)^{-1}$, then $\|(\lambda I+A)^{-1}\|\le\frac{K^*}{|\lambda|}$. So we proved (ii).

3. Main Result and its Proof

In this section, we state and prove the following main conclusion of this paper.

Theorem *Let $f(x,t,p)\in C(\Omega\times R^+\times R^1)$ and $f(x,t,p)$ be bounded. Assume that for any $u\in H_0^1(\Omega)$, $f(\cdot,t,u)\in C([0,T];L^2(\Omega))$ and the following conditions be satisfied*

$$\text{(i) } \|f(\cdot,t,u_1)-f(\cdot,t,u_2)\|_0 \le N\|u_1-u_2\|_0. \tag{3.1}$$

Where N is a positive constant;

$$\text{(ii) } u_0(x_0)=r(0), \qquad \text{for any } u_0\in H_0^1(\Omega). \tag{3.2}$$

$$\text{(iii) } \varphi(t)\in C^1([0,T]), \varphi(t)\ne 0, t\in[0,T]. \tag{3.3}$$

Then the inverse problem $\{(1.1)$–$(1.5)\}$ has a unique solution $\{u(x,t),p(x,t)\}$ as $T>0$ is an appropriate small number.

To prove this theorem, we observe and study an equivalent integral equation. we assume that $S(t)$ be the C_0-contractive semigroup with the generator $-A=-L^{-1}b\Delta=$

$-(I-\Delta)^{-1}b\Delta$. From lemma 2, Cauchy problem (2.1) and (2.2) can be returned to the following Volterra integral

$$u(t)=S(t)u_0+\int_0^t S(t-\tau)L^{-1}f(x,\tau,u(\tau))\,\mathrm{d}\tau+\int_0^1 S(t-\tau)L^{-1}\varphi(\tau)p(x,\tau)\,\mathrm{d}\tau. \quad (3.4)$$

For each $u\in H_0^1(\Omega)$, we take the derivative for t on both sides of (3.4). Note the fundamental properties $S(0)=I$, $S(t)u_0$ is the solution of the homogeneous Cauchy problem[4,6], and make use of the additional condition (1.5), we can deduce

$$\begin{aligned} r'(t)=\frac{\mathrm{d}u(t)}{\mathrm{d}t}&=-S(t)Au_0+L^{-1}f(x,t,u(t))+L^{-1}\varphi(t)p(x,t)-\\ &\int_0^t S(t-\tau)AL^{-1}f(x,\tau,u(\tau))\,\mathrm{d}\tau-\int_0^t S(t-\tau)AL^{-1}\varphi(\tau)p(x,\tau)\,\mathrm{d}\tau. \end{aligned} \quad (3.5)$$

Putting the differential operator $L\varphi^{-1}(t)$ on both sides of (3.5), then we get

$$\begin{aligned} &L\varphi^{-1}(t)(r'(t)+S(t)Au_0)-L\varphi^{-1}(t)L^{-1}f(x,t,u(t))+\\ &\quad L\varphi^{-1}(t)\int_0^t S(t-\tau)AL^{-1}f(x,\tau,u(\tau))\,\mathrm{d}\tau+\\ &\quad L^{-1}\varphi^{-1}(t)\int_0^t S(t-\tau)AL^{-1}\varphi(\tau)p(x,\tau)\,\mathrm{d}\tau\\ &=p(x,t)\overset{\mathrm{def}}{=}p(t). \end{aligned} \quad (3.6)$$

To simplify the expression of $p(t)$ in (3.6), we introduce some symbols of the following differential operators

$$p_0(t)=L\varphi^{-1}(t)(r'(t)+S(t)Au_0),\ \varphi_1(t)=-\varphi^{-1}(t),\ \varphi_2(t,\tau)=A\varphi_1(t)S(t-\tau),$$
$$\varphi_3(t,\tau)=\varphi_2(t,\tau)\varphi(\tau),\ Q_0(t)=S(t)u_0,\ \varphi_4(t,\tau)=S(t-\tau)L^{-1},$$
$$\varphi_5(t,\tau)=\varphi_4(t-\tau)\varphi^{-1}(t).$$

Therefore, the inverse problem $\{(2.1),(2.2),(1.5)\}$ can be changed into the following equivalent integral equations

$$p(t)=p_0(t)+\varphi_1(t)f(x,t,u(t))+\int_0^t\varphi_2(t,\tau)f(x,\tau,u(\tau))\mathrm{d}\tau+\int_0^t\varphi_3(t,\tau)p(\tau)\mathrm{d}\tau \quad (3.7)$$

and

$$f(x,t,u(t))=f(x,t,Q_0(t)+\int_0^t\varphi_4(t,\tau)f(x,\tau,u(\tau))\mathrm{d}\tau+\int_0^t\varphi_5(t,\tau)p(\tau)\mathrm{d}\tau). \quad (3.8)$$

Denote $\mathcal{D}=\{(p,f)\mid(p,f)\in C([0,T];L^2(\Omega)\times L^2(\Omega))\}$, and for any $(p,f)\in\mathcal{D}$, define a norm $\|(p,f)\|=\max\limits_{t\in[0,T]}\{\|p\|_0,\|f\|_0\}$. With the exception of this, we define a mapping $\mathcal{A}:(p,f)|\to(\bar{p},\bar{f})$ for $(p,f)\in\mathcal{D}$. Where $\bar{p}$ and $\bar{f}$ are determined by the right terms of (3.7) and (3.8) respectively.

It is easy to know $\mathcal{A}$ is a mapping of $\mathcal{D}$ onto $\mathcal{D}$. Notice that $S(t)$ is a C_0-contractive semigroup, then we see, for any $t \in [0,T], L^{-1} = (I-\Delta)^{-1}$ is a bounded linear operator. Note $M = \max_{(t,\tau)\in[0,T]\times[0,T]}\{\|\varphi_1(t)\|, \|\varphi_i(t,\tau)\|, i = 2,3,4,5\}$, then for $v_1 = (p_1, f_1), v_2 = (p_2, f_2) \in \mathcal{D}$, by the condition (3.1), there is

$$\|\bar{f}_1(t) - \bar{f}_2(t)\| \leq 2MNT\|v_1 - v_2\|. \tag{3.9}$$

Put (3.8) into (3.7), we have

$$\begin{aligned} p(t) =& p_0(t) + \varphi_1(t)f(x,t,Q_0(t) + \int_0^t \varphi_4(t,\tau)f(x,\tau,u(\tau))\mathrm{d}\tau + \int_0^t \varphi_5(t,\tau)p(\tau)\mathrm{d}\tau)+ \\ & \int_0^t \varphi_2(t,\tau)f(x,\tau,u(\tau))\mathrm{d}\tau + \int_0^t \varphi_3(t,\tau)p(\tau)\mathrm{d}\tau. \end{aligned}$$

Similar to (3.9), there is

$$\|\bar{p}_1(t) - \bar{p}_2(t)\| \leq 2MT(MN+1)\|v_1 - v_2\|, \tag{3.10}$$

where $(\bar{p}_i, \bar{f}_i) = \mathcal{A}(p_i, f_i) = \mathcal{A}v_i, i = 1,2$. From (3.9) and (3.10), we can infer that

$$\|\mathcal{A}v_1 - \mathcal{A}v_2\| \leq 2MN^*T\|v_1 - v_2\|.$$

The above inequality shows that as $T < \frac{1}{2MN^*}$, the mapping $\mathcal{A}$ is constractive. So by the fixed point principle, there exists a unique fixed point v, such that $\mathcal{A}v = v$. This means inverse problem $\{(2.1),(2.2),(1.5)\}$ (i.e., inverse problem $\{(1.1)$–$(1.5)\}$) has a unique solution $\{u(x,t), p(x,t)\}$.

The proof of the solvability of the inverse problem $\{(1.1)$–$(1.5)\}$ is completed.

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一非线性发展方程的反问题

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摘 要: 研究一非线性发展方程的未知源项的反演问题. 首先，把所考虑的初边值问题化成一等价非线性发展方程的 Cauchy 问题；然后，利用半群理论，论证反问题解的存在性和唯一性；最后，利用压缩映射不动点方法，得到反问题的可解性.