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Meromorphic Functions Share One Value with Their Derivatives *

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Abstract: In this paper we will improve Brück's result and give other results.

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1. Introduction

In this paper a "meromorphic function" will mean that is meromorphic in the whole complex plane. We say that two non-constant meromorphic functions f and g share a value c in the extended complex plane provided that $f(z_0) = c$ if and only if $g(z_0) = c$. We will state weather a share value is by CM (counting multiplicitics) or by IM (ignoring multiplicities). We denote $\bar{E}_{k)}(c,f)$ the set of zeros of $f(z)-c$ with multiplicities less then or equal to k (ignoring multiplicity), $N_{k)}(\frac{1}{f-c})$ denotes the counting function of c-points of f with mulliplicities less than or equal to k and $N_2(r,\frac{1}{f-c})$ denotes the counting function of c-points of f, where a p-fold c-point is connted with multiplicity min(p,2). Finally we will use the standard notations and results of the Nevanlinna theory (see [1] or [4] for example).

In [2] Mues and Steinmetz proved the following:

Theorem A *Let f be an entire function which is not constant. If f and f' share the two distinct finite complex numbers, then $f \equiv f'$.*

In general, this theorem is false if f and f' share only one value. This may be seen by the example $f(z) = e^{e^z}\int_0^z e^{-e^t}(1-e^t)\mathrm{dt}$. In this case, we have $f'-1 = e^z(f-1)$, so that f and f' share the value 1 (even CM), but $f \not\equiv f'$.

In 1996 R.Brück [3] proved the following:

Theorem B *Let f be an entire function which is not constant. If f and f' share the value 1 CM, and if $N(r,\frac{1}{f'}) = S(r,f)$, then*

$$f-1 = c(f'-1),$$

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where $c \in C\backslash\{0\}$.

2. Lemmas

Lemma 1 *Let f be a non-constant meromorphic function, k be a positive integer, then*

$$N(r, \frac{1}{f^{(k)}}) \leq N(r, \frac{1}{f}) + k\bar{N}(r, f) + S(r, f).$$

Proof See [4, P. 41]

Lemma 2 *Let f be a non-constant meromorphic function. Then for any $k \geq 1$ we have*

$$T(r, f) \leq N(r, \frac{1}{f-b}) + \bar{N}(r, \frac{1}{f^{(k)}-a}) + \bar{N}(r, f) + S(r, f),$$

where $a(\neq 0)$ and b are constants.

Proof Cleary

$$\begin{aligned} m(r, \frac{1}{f-b}) + m(r, \frac{1}{f^{(k)}-a}) &\leq m(r, \frac{1}{f^{(k)}}) + m(r, \frac{1}{f^{(k)}-a}) + S(r, f) \\ &\leq m(r, \frac{1}{f^{k+1}}) + S(r, f). \end{aligned}$$

And hence

$$\begin{aligned} T(r, f) + T(r, f^{(k)}) \leq & N(r, \frac{1}{f-b}) + N(r, \frac{1}{f^{(k)}-a}) - N(r, \frac{1}{f^{(k+1)}}) + \\ & T(r, f^{(k+1)}) + S(r, f). \end{aligned} \tag{2.1}$$

By Milloux inequality (see [1] or [4] we have

$$T(r, f^{(k+1)}) \leq T(r, f^{(k)}) + \bar{N}(r, f) + S(r, f). \tag{2.2}$$

Clearly

$$N(r, \frac{1}{f^{(k)}-a}) - N(r, \frac{1}{f^{(k+1)}}) \leq N(r, \frac{1}{f^{(k)}-a}). \tag{2.3}$$

From (2.1),(2.2) and (2.3) we finshed the proof of lemma 2 □

Lemma 3 *Let f be a non-constant entire function and f and f' share the value 1 CM. If $f - 1 \neq c(f'-1), c \in C\backslash\{0\}$, then*

$$N(r, \frac{1}{f'}) = N_{1)}(r, \frac{1}{f'}) + S(r, f).$$

Proof Set

$$H = \frac{f'}{f-1} - \frac{f''}{f'-1}. \tag{2.4}$$

From the fundamental estimate of the logarithmic derivative it follows that

$$m(r,H)=S(r,f). \tag{2.5}$$

Since f and f' that share 1 CM, therefore $N(r,H)\equiv 0$. Form this and (2.5) we find

$$T(r,H)=S(r,f). \tag{2.6}$$

If $H\equiv 0$, then $f-1=c(f'-1), c\in C\backslash\{0\}$. This is contradiction. We now suppose $H\not\equiv 0$. Since f and f' that share 1 CM, therefore all zeros of $f-1$ are simple, from this and (2.4), (2.6) we find $N_{(2}(r,\frac{1}{f'})\leq 2N(r,\frac{1}{H})\leq 2T(r,H)+O(1)=S(r,f)$. From this we finished proof of lemma 2. □

3. Theorems

Theorem 1 *Let f be a non-constant meromorphic function, and $\bar{E}_{n)}(1,f)=\bar{E}_{n)}(1,f^{(k)})$, n is a positive integer or $\infty, k\geq 1$. If*

$$2\bar{N}(r,f)+N(r,\frac{1}{f^{(k)}})+\bar{N}(r,\frac{1}{f'})+\bar{N}_{(2}(r,\frac{1}{f^{(k)}})+N_{(2}(r,\frac{1}{f-1})$$
$$<(\lambda+0(1))T(r,f^{(k)}),\quad (r\to\infty, r\notin E),$$

for some real number $\lambda\in(0,1)$, where E is a set of finite measure. Then

$$f-1=c(f^{(k)}-1),$$

where $c\in C\backslash\{0\}$.

Proof Set

$$\Delta=\frac{f^{(k+2)}}{f^{(k+1)}}-\frac{f''}{f'}+2\frac{f'}{f-1}-2\frac{f^{(k+1)}}{f^{(k)}-1}. \tag{3.1}$$

From the fundamental estimate of the logarithmic derivative it follows that

$$m(r,\Delta)=S(r,f). \tag{3.2}$$

Since $\bar{E}_{n)}(1,f)=\bar{E}_{n)}(1,f^{(k)})$, therefore from (3.1) we find

$$N(r,\Delta)\leq\bar{N}(r,\frac{1}{f})+\bar{N}(r,\frac{1}{f^{(k+1)}})+\bar{N}(r,f).$$

From this and (3.2) we get

$$T(r,\Delta)\leq\bar{N}(r,\frac{1}{f'})+N(r,\frac{1}{f^{(k+1)}})+\bar{N}S(r,f)+S(r,f). \tag{3.3}$$

Let z_0 be a simple zero for $f-1$ and $f^{(k)}-1$, then from (3.1) we get $\Delta(z_0)=0$. Thus if $\Delta\not\equiv 0$, then from (3.1) we find

$$N_{1)}(r,\frac{1}{f^{(k)}}-1)\leq N(r,\frac{1}{\Delta})+\bar{N}_{(2}(r,\frac{1}{f-1})$$
$$\leq T(r,\Delta)+\bar{N}_{(2}(r,\frac{1}{f-1})+O(1).$$

From this and (3.3) we get

$$N_{1)}(r, \frac{1}{f^{(k)}-1}) \leq \bar{N}(r, \frac{1}{f'}) + \bar{N}(r, \frac{1}{f^{(k+1)}}) + \bar{N}(r, f) + \bar{N}_{(2}(r, \frac{1}{f-1}) + S(r, f).$$

From this and the second fundamental theorem we find

$$T(r, f^{(k)}) \leq N(r, \frac{1}{f^{(k)}}) + N_{(2}(r, \frac{1}{f^{(k)}-1}) + \bar{N}(r, \frac{1}{f'}) + \bar{N}_{(2}(r, \frac{1}{f-1}) + 2\bar{N}(r, f) + S(r, f).$$

This is contradiction with condition of Theorem 1. Therefore $\Delta \equiv 0$ and hence by integrating of (3.1) we get

$$c\frac{f^{(k+1)}}{f'} = (\frac{f^{(k)}-1}{f-1})^2, \tag{3.4}$$

where c is a nonzero constant.

Again integrating of (3.4) we find $\frac{1}{f^{(k)}-1} = \frac{a}{f-1} + b$, where $a(\neq 0)$ and b are constants. From this easy to see f and $f^{(k)}$ share the value 1 CM.

We assume $\frac{f^{(k+1)}}{f'}$ is not constant. From (3.4) we find

$$N(r, \frac{f^{(k+1)}}{f'}) = S(r, f). \tag{3.5}$$

Clearly from (3.4) we find

$$\begin{aligned} N_{1)}(r, \frac{1}{f-1}) &\leq N(r, \frac{1}{\frac{f^{(k+1)}}{f'} - c}) \leq T(r, \frac{f^{(k+1)}}{f'}) + O(1) \\ &\leq N(r, \frac{f^{(k+1)}}{f'}) + S(r, f) \\ &\leq S(r, f). \end{aligned}$$

And hence $N_{1)}(r, \frac{1}{f^{(k)}-1}) = S(r, f)$. From this and the second fundamental theorem for $f^{(k)}$ we find

$$T(r, f^{(k)}) \leq \bar{N}(r, \frac{1}{f^{(k)}}) + \bar{N}_{(2}(r, \frac{1}{f^{(k)}-1}) + \bar{N}(r, f) + S(r, f).$$

This is contradiction with condition of Theorem 1. Therefore $\frac{f^{(k+1)}}{f'}$ is a constant, and so the proof of Theorem 1 is finished. □

Corollary 1 *Let f be a non-constant meromorphic function, and $\bar{E}_{n)}(1, f) = \bar{E}_{n)}(1, f^{(k)})$, n is a positive integer or $\infty, k \geq 1$. If*

$$N(r, \frac{1}{f'}) + \bar{N}(r, f) = S(r, f). \tag{3.6}$$

Then

$$f - 1 = c(f^{(k)} - 1),$$

where $c \in C\backslash\{0\}$.

Proof From (3.6) and lemma 1 we get $N(r, \frac{1}{f^{(k+1)}}) = S(r,f)$. From this and (3.6) we find

$$N_{(2}(r, \frac{1}{f^{(k)}-1}) = S(r,f) \text{ and } N_{(2}(r, \frac{1}{f-1}) = S(r,f). \tag{3.7}$$

Again by lemma 1 and (3.6) we find

$$N(r, \frac{1}{f^{(k)}}) \leq N(r, \frac{1}{f'}) + (k-1)\bar{N}(r,f) = S(r,f).$$

From this and (3.7) we find the conditions of Theorem 1 is statisfied. Therefore $f - 1 = c(f^{(k)} - 1), c \in C\backslash\{0\}$.□

Remark 1 The condition (3.6) in corollary 1 is necessary condition. For example. Let

$$f(z) = \frac{(1-i)e^{2iz} + (1+i)b}{e^{2iz} + b}, \tag{3.8}$$

where $b \in C\backslash\{0\}$. Clearly f and f' share the value 1 IM and,

$$N(r, \frac{1}{f'}) + \bar{N}(r,f) = T(r,f) + S(r,f) \neq S(r,f).$$

But $f - 1 \neq c(f' - 1), c \in C\backslash\{0\}$.

In Corollary 1, if $n = \infty$, then the condition $\bar{E}_{n)}(1,f) = \bar{E}_{n)}(1,f^{(k)})$ becomes $f, f^{(k)}$ share the value 1 IM and if f is entier function, then $\bar{N}(r,f) \equiv 0$. Thus we have the following corollary :

Corolary 2 *Let f be non-constant entire function and $f, f^{(k)}$ share the value 1 IM and if, $N(r, \frac{1}{f'}) = S(r,f)$,then*

$$f - 1 = c(f^{(k)} - 1),$$

where $c \in C\backslash\{0\}$.

Remark 2 Clearly the corollary 2 is improve the Theorem A.

Corollary 3 *Let f be a non-constant entire function, and f, f' share the value 1 CM. If*

$$N_{1)}(r, \frac{1}{f'}) < (\lambda + o(1))T(r,f'), \quad (r \to \infty, r \notin E),$$

for some real number $\lambda \in (0, \frac{1}{2})$, where E is a set of finite measure. Then

$$f - 1 = c(f' - 1), \tag{3.9}$$

where $c \in C\backslash\{0\}$.

Proof Since f and f' share the value 1 CM, therefore all the zeros of $f-1$ and $f'-1$ are simple, and hence

$$N_{(2}(r, \frac{1}{f'-1}) = N_{(2}(r, \frac{1}{f-1}) \equiv 0. \tag{3.10}$$

From (3.10), lemma 3 and Theorem 1 can be get immediately (3.9).□

Remark 3 Clearly $N_{1)}(r,\frac{1}{f'}) \le N(r,\frac{1}{f'})$, therefore the corollary 3 is also improve the Theorem A.

Corollary 4 *Let f be a non-constant meromorphic function, and f and f' share the value 1 IM. If*

$$4\bar{N}(r,f)+3\bar{N}(r,\frac{1}{f'})+N(r,\frac{1}{f'})<(\lambda+0(1))T(r,f'),\ (r\to\infty, r\notin E),$$

for some real number $\lambda\in(0,1)$, where E is a set of finite measure. then (3.9) is satisfies.

Proof If $\frac{f''}{f'}$ is a constant, then $f'=c_1f+c_2$, where $c_1(\ne 0)$ and c_2 are constants. From lemma 2 we see $N(r,\frac{1}{f-1})\ne S(r,f)$. Therefore $c_1+c_2=1$ and hence (3.9) is satisfied.

We now suppose $\frac{f''}{f'}$ is a non-constant.Clearly

$$N_{(2}(r,\frac{1}{f'-1})\le 2N(r,\frac{1}{\frac{f''}{f'}})\le 2T(r,\frac{f''}{f'})+O(1)$$
$$\le 2\bar{N}(r,\frac{1}{f'})+2\bar{N}(r,f).$$

From this and Theorem 1 can be get immediately (3.9). □

Theorem 2 *Let f be a non-constant meromorphic function and f,f' share the value 1 IM.If*

$$N_2(r,\frac{1}{f'})+N_2(r,\frac{1}{f})+3\bar{N}(r,f)+N_{(2}(r,\frac{1}{f'-1})-m(r,\frac{1}{f'-1})$$
$$<(\lambda+o(1))T(r,f),\quad (r\to\infty, r\notin E),$$

for some real number $\lambda\in(0,1)$, where E is a set of finite measure. Then $f\equiv f'$.

Proof From (3.1)(take $k=1$) we find

$$N(r,\Delta)\le \bar{N}_{(2}(r,\frac{1}{f})+\bar{N}_{(2}(r,\frac{1}{f'})+\bar{N}_{(2}(r,\frac{1}{f'-1})+\bar{N}(r,f)+N_0(r,\frac{1}{f'})+N_0(r,\frac{1}{f''}),\quad (3.11)$$

where $N_0(r,\frac{1}{f'})$ refers the counting function of the zeros of f' which is not come from f or $f-1$, and $N_0(r,\frac{1}{f''})$ is defined analogusly.Since f and f' share the value 1 IM,therefore

$$\bar{N}(r,\frac{1}{f-1})+\bar{N}(r,\frac{1}{f'-1})=2\bar{N}(r,\frac{1}{f'-1})<N_{1)}(r,\frac{1}{f'-1})+N(r,\frac{1}{f'-1}).\quad (3.12)$$

If $\Delta\not\equiv 0$, then (3.1) we find

$$N_{1)}(r,\frac{1}{f'-1})\le N(r,\frac{1}{\Delta})\le T(r,\Delta)+O(1).\quad (3.13)$$

From (3.2),(3.11),(3.12) and (3.13) we get

$$2\bar{N}(r,\frac{1}{f'-1}) < N(r,\frac{1}{f'-1}) + \bar{N}_{(2}(r,\frac{1}{f}) + \bar{N}_{(2}(r,\frac{1}{f'}) + \bar{N}_{(2}(r,\frac{1}{f'-1}) + \bar{N}(r,f) + N_0(r,\frac{1}{f'}) + N_0(r,\frac{1}{f''}) + S(r,f). \quad (3.14)$$

From the second fundamental theorem for f and f' we have

$$T(r,f) + T(r,f') \leq \bar{N}(r,\frac{1}{f}) + \bar{N}(r,\frac{1}{f'}) + 2\bar{N}(r,\frac{1}{f'-1}) + 2\bar{N}(r,f) - N_0(r,\frac{1}{f'}) - N_0(r,\frac{1}{f''}) + S(r,f).$$

From this and (3.14) we find

$$T(r,f) + T(r,f') \leq \bar{N}(r,\frac{1}{f}) + \bar{N}_{(2}(r,\frac{1}{f}) + \bar{N}(r,\frac{1}{f'}) + \bar{N}_{(2}(r,\frac{1}{f'}) + 3\bar{N}(r,f) + \bar{N}_{(2}(r,\frac{1}{f'-1}) + N(r,\frac{1}{f'-1}) + S(r,f).$$

And hence

$$T(r,f) < N_2(r,\frac{1}{f}) + N_2(r,\frac{1}{f'}) + 3\bar{N}(r,f) + N_{(2}(r,\frac{1}{f'-1}) - m(r,\frac{1}{f'-1}) + S(r,f).$$

This is contradiction with condition in Theorem 2.Therefore $\Delta \equiv 0$. The same way as Theorem 1 we get $f-1 = c(f'-1)$, where c is a nonzero constant.

By solve this equation we find

$$f(z) = Ae^{\frac{1}{c}z} - (c-1), \quad (3.15)$$

where A is a nonzero constant.

If $c \neq 1$, then from (3.15) we see $T(r,f) = N(r,\frac{1}{f}) + S(r,f)$. From the condition of Theorem 2 we get $T(r,f) = S(r,f)$. This is impossible. Therefore $c = 1$ and hence $f \equiv f'$.

Remark 4 The condition in Theorem 2 is necessary condition see examples (3.15) and (3.8).

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