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Meromorphic Solutions of Systems of Higher-Order Algebraic Differential Equations in Complex Domain *

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Abstract: Using the Nevanlinna theory of the value distribution of meromorphic functions,we investigate the problem of the existence of admissible meromorphic solutions of a type of systems of higher-order algebraic differential equations.

Key words: meromorphic solutions; systems of differential equations; the value distribution.

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1. Introduction and main result

We use the standard notation of Nevanlinna theory (see e.g.[1]). The problem of the existence of admissible solutions of systems of algebraic differential equations have been studied by several authors and some results have been obtained(see [2],[3],[4],[6-9]).

In this paper we will mainly consider the following systems of higher-order algebraic differential equations

$$\begin{cases} \Omega_1(z,w_1,w_2)/(w_2-b)^{\lambda_2} = \sum_{i=0}^{p_1} a_i(z)w_1^i / \sum_{j=0}^{q_1} b_j(z)w_1^j, \\ \Omega_2(z,w_1,w_2)/(w_1-a)^{\overline{\lambda}_1} = \sum_{i=0}^{p_2} c_i(z)w_2^i / \sum_{j=0}^{q_2} d_j(z)w_2^j, \end{cases} \tag{1.1}$$

where

$$\Omega_1(z,w_1,w_2) = \sum_{(i)} a_{(i)}(z) \prod_{k=1}^{2} (w_k)^{i_{k0}} (w_k')^{i_{k1}} \dots (w_k^{(n)})^{i_{kn}}, \tag{1.2}$$

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$$\Omega_2(z,w_1,w_2)=\sum_{(j)} b_{(j)}(z)\prod_{k=1}^{2}(w_k)^{j_{k0}}(w_k')^{j_{k1}}\ldots(w_k^{(n)})^{j_{kn}} \tag{1.3}$$

are differential polynomials,$(t)=(t_{10},t_{20},\ldots,t_{1n},t_{2n}),(t=i,j)$ are multi-indices of non-negative integers,both a and b are nonzero constants,$\{a_{(i)}(z)\},\{b_{(j)}(z)\}$,$\{a_i(z)\},\{b_j(z)\}$, $\{c_i(z)\}$ and $\{d_j(z)\}$ are meromorphic functions, $\lambda_2,\overline{\lambda}_1$ are as in Definition 1.

Definition 1 *For (1.1)-(1.3), write*

$$\lambda_k=\max_{(i)}\{\sum_{l=0}^{n} i_{kl}\}, u_k=\max_{(i)}\{\sum_{l=0}^{n} l i_{kl}\}, \Delta_k=\max_{(i)}\{\sum_{l=0}^{n}(l+1)i_{kl}\};$$

$$\overline{\lambda}_k=\max_{(j)}\{\sum_{l=0}^{n} j_{kl}\}, \overline{u}_k=\max_{(j)}\{\sum_{l=0}^{n} l j_{kl}\}, \overline{\Delta}_k=\max_{(j)}\{\sum_{l=0}^{n}(l+1)j_{kl}\}; k=1,2.$$

$$S_1(r)=\sum_{(i)}T(r,a_{(i)})+\sum_{(j)}T(r,b_{(j)})+\sum_{i=0}^{p_1}T(r,a_i)+\sum_{j=0}^{q_1}T(r,b_j)+\sum_{i=0}^{p_2}T(r,c_i)+\sum_{j=0}^{q_2}T(r,d_j).$$

Definition 2 *Let $(w_1(z),w_2(z))$ be a meromorphic solution of (1.1) .Suppose the component w_i of (w_1,w_2) satisfies the following condition:$S_1(r)=o(T(r,w_i)),i=1,2,r\notin I_1$. We say that w_i is an admissible component of solutions of (1.1),where I_1 is a set of finite linear measure.*

We get the following Theorem:

Theorem 1 *Let $(w_1(z),w_2(z))$ be a meromorphic solution of (1.1).If the components w_1 and w_2 are admissible,then $q_1q_2\le[\lambda_2+u_2(1-\theta(w_2,\infty))][\overline{\lambda}_1+\overline{u}_1(1-\theta(w_1,\infty))]$, where $\theta(w_k,\infty)=1-\limsup\overline{N}(r,w_k)/T(r,w_k)$ for $k=1,2$.*

Corollary 1 *Let $(w_1(z),w_2(z))$ be an entire solution of (1.1) .If the components w_1 and w_2 are admissible,then $q_1q_2\le\lambda_2\overline{\lambda}_1$.*

The proof of corollary 1 is obvious.

2. Preliminary Lemmas

Lemma 1 *Let $\Omega_1(z,w_1,w_2)$ be as (1.2),$Q_{p_1}(z,w_1)=\sum_{i=0}^{p_1}a_i(z)w_1^i$, $Q_{q_1}(z,w_1)=\sum_{j=0}^{q_1}b_j(z)w_1^j$,$b$ be a nonzero constant, $\{a_{(i)}(z)\},\{a_i(z)\}$ and $\{b_j(z)\}$ are meromorphic functions.If*

$$Q_{q_1}(z,w_1)\frac{\Omega_1(z,w_1,w_2)}{(w_2-b)^{\lambda_2}}=Q_{p_1}(z,w_1), q_1>p_1, \tag{2.1}$$

then

$$\begin{aligned} m(r,\tfrac{\Omega_1(z,w_1,w_2)}{(w_2-b)^{\lambda_2}})\le\ & \lambda_2 m(r,\tfrac{1}{w_2-b})+\sum_{(i)}m(r,a_{(i)})+\sum_i m(r,a_i)+\sum_j m(r,b_j)\\ +\ & O\{\sum_{i=1}^{n}m(r,\tfrac{w_1^{(i)}}{w_1}+\sum_{i=1}^{n}m(r,\tfrac{(w_2-b)^{(i)}}{w_2-b})\}+O(1). \end{aligned} \tag{2.2}$$

Proof We rewrite $\Omega_1(z,w_1,w_2)/(w_2-b)^{\lambda_2}$ as follows:

$$\frac{\sum_{(i)} a_{(i)}(z)\prod_{k=1}^{2}(w_k)^{i_{k0}}(w'_k)^{i_{k1}}\dots(w_k^{(n)})^{i_{kn}}}{(w_2-b)^{\lambda_2}}=\frac{\sum_{(i)} a_{(i)}(z)w_1^{i_{10}+\dots+i_{1n}}\prod_{k=1}^{n}(\frac{w_1^{(k)}}{w_1})^{i_{1k}}(\frac{(w_2-b)^{(k)}}{w_2-b})^{i_{2k}}}{(w_2-b)^{\lambda_2-\sum_{l=0}^{n} i_{2l}}}.$$

Let $E_r=\{z:|z|=r\}, E_1=\{z\in E_r:|w_1|<1\}, E_2=E_r-E_1$.

If $z\in E_1$,then $|\frac{\Omega_1(z,w_1,w_2)}{(w_2-b)^{\lambda_2}}|\le\max\{1,(\frac{1}{w_2-b})^{\lambda_2}\}\sum_{(i)}|a_{(i)}(z)|\prod_{k=1}^{n}|\frac{w_1^{(k)}}{w_1}|^{i_{1k}}|\frac{(w_2-b)^{(k)}}{w_2-b}|^{i_{2k}}$, we have

$$\begin{aligned}&\tfrac{1}{2\pi}\int_{E_1}\log^+|\tfrac{\Omega_1(z,w_1,w_2)}{(w_2-b)^{\lambda_2}}|d\theta\le\tfrac{\lambda_2}{2\pi}\int_{E_1}\log^+|\tfrac{1}{w_2-b}|\mathrm{d}\theta+\tfrac{1}{2\pi}\sum_{(i)}(\int_{E_1}\log^+|a_{(i)}(z)|\mathrm{d}\theta\\&+\sum_{k=1}^{n}i_{2k}\int_{E_1}\log^+|\tfrac{(w_2-b)^{(k)}}{w_2-b}|\mathrm{d}\theta+\sum_{k=1}^{n}i_{1k}\int_{E_1}\log^+|\tfrac{(w_1)^{(k)}}{w_1}|\mathrm{d}\theta)+O(1).\end{aligned}\tag{2.3}$$

If $z\in E_2$,then $Q_{q_1}(z,w_1)=b_{q_1}(z)(w_1^{q_1}+A_1(z)w_1^{q_1-1}+\dots+A_{q_1}(z))$, where $A_i(z)=b_{q_1-i}(z)/b_{q_1}(z)$ and for each $z\in\mathbf{C}$ we define $A(z)=\max_{1\le i\le q_1}\{1,|A_i(z)|^{1/i}\}$.

Put $E_3=\{z\in E_r,|w_1|\le 2A(z)\}, E_{21}=E_2\cap E_3, E_{22}=E_2-E_{21}$. We have for $z\in E_{21}$

$$|\frac{\Omega_1(z,w_1,w_2)}{(w_2-b)^{\lambda_2}}|\le(2A(z))^{\lambda_1}\sum_{(i)}|a_{(i)}(z)||(\frac{1}{w_2-b})^+|^{\lambda_2}\prod_{k=1}^{n}|\frac{w_1^{(k)}}{w_1}|^{i_{1k}}|\frac{(w_2-b)^{(k)}}{w_2-b}|^{i_{2k}}.$$

Note that $\log^+A(z)\le\log^+(1+A_1(z)+\dots+A_{q_1}(z)^{1/q_1})\le\sum_{j=1}^{q_1}\log+|b_{q_1-j}|+q_1\log|\frac{1}{b_{q_1}}|+O(1)$. Thus,we obtain

$$\begin{aligned}&\frac{1}{2\pi}\int_{E_{21}}\log^+|\frac{\Omega_1(z,w_1,w_2)}{(w_2-b)^{\lambda_2}}|\mathrm{d}\theta\le\frac{\lambda_2}{2\pi}\int_{E_1}\log^+|\frac{1}{w_2-b}|\mathrm{d}\theta+\frac{\lambda_1}{2\pi}(\sum_{j=1}^{q_1}\int_{E_{21}}\log^+|b_{q_1-j}|\mathrm{d}\theta+\\&q_1\int_{E_{21}}\log^+|\frac{1}{b_{q_1}}|\mathrm{d}\theta)+\frac{1}{2\pi}\sum_i(\int_{E_{21}}\log^+|a_{(i)}(z)|\mathrm{d}\theta+\sum_{k=1}^{n}i_{2k}\int_{E_{21}}\log^+|\frac{(w_2-b)^{(k)}}{w_2-b}|\mathrm{d}\theta+\\&\sum_{k=1}^{n}i_{1k}\int_{E_{21}}\log^+|\frac{(w_1)^{(k)}}{w_1}|\mathrm{d}\theta)+O(1).\end{aligned}\tag{2.4}$$

For $z\in E_{22}$,we have

$$\begin{aligned}|Q_{q_1}(z,w_1)|=&|b_{q_1}(z)||w_1|^{q_1}|1+\sum_{i=1}^{q_1}\frac{A_i(z)}{w_1^i}|\ge|b_{q_1}(z)||w_1|^{q_1}\{1-\sum_{i=1}^{q_1}\frac{|A_i(z)|}{|w_1|^i}\}\\\ge&|b_{q_1}(z)||w_1|^{q_1}\{1-\sum_{i=1}^{q_1}\frac{|A(z)|^i}{|2A(z)|^i}\}=\frac{|b_{q_1}(z)||w_1|^{q_1}}{2^{q_1}}.\end{aligned}$$

It follows that

$$|\frac{\Omega_1(z,w_1,w_2)}{(w_2-b)^{\lambda_2}}|=\frac{|Q_{p_1}(z,w_1)|}{|Q_{q_1}(z,w_1)|}\le\frac{2^{q_1}\sum_{i=0}^{p_1}|a_i(z)||w_1|^{i-q_1}}{|b_{q_1}(z)|}\le\frac{2^{q_1}\sum_{i=0}^{p_1}|a_i(z)|}{|b_{q_1}(z)|},$$

we get

$$\frac{1}{2\pi}\int_{E_{22}}\log^+|\frac{\Omega_1(z,w_1,w_2)}{(w_2-b)^{\lambda_2}}|d\theta \le \frac{\lambda_2}{2\pi}\int_{E_{22}}\log^+|\frac{1}{|b_{q_1}|}|d\theta+\sum_{i=0}^{p_1}\int_{E_{22}}\log^+|a_i(z)|d\theta. \quad (2.5)$$

Combining (2.3),(2.4) and (2.5),we get the inequality (2.2).

This completes the proof of Lemma 1.

Lemma 2 *Let the conditions be as Lemma 2.1.Then*

$$N(r,\frac{\Omega_1(z,w_1,w_2)}{(w_2-b)^{\lambda_2}}) \le \lambda_2 N(r,\frac{1}{w_2-b})+u_2\overline{N}(r,w_2)+\sum_{(i)}N(r,a_{(i)})+ \sum_{i=0}^{p_1}N(r,a_i)+O\{\sum_{j=0}^{q_1}N(r,\frac{1}{b_j})\}. \quad (2.6)$$

Proof The poles of $\frac{\Omega_1(z,w_1,w_2)}{(w_2-b)^{\lambda_2}}$ may arise only from one of the following cases:

(I) The zeros of w_2-b

(II) The poles of w_1 but not any zero of w_2-b

(III) The poles of w_2 but not any pole of w_1.

Case (I) z_0 is a zero of w_2-b.

(a) If z_0 is not a b value point of w_1,then

$$\tau(z_0,\Omega_1(z,w_1,w_2)/(w_2-b)^{\lambda_2}) \le \lambda_2\tau(z_0,1/(w_2-b))+\sum_{(i)}\tau(r,a_{(i)}). \quad (2.7)$$

(b) If z_0 is a b value point of w_1,then

$$\tau(z_0,\Omega_1(z,w_1,w_2)/(w_2-b)^{\lambda_2}) \le \max\{0,\lambda_2-\lambda_1\}\tau(z_0,1/(w_2-b))+\sum_{(i)}\tau(r,a_{(i)}). \quad (2.8)$$

Case (II) (a) If z_0 is a pole of w_1 with multiplicity τ but any poly and zero of coefficients,then z_0 is a pole of the left side of (2.1),its multiplicity is at least τq_1;On the other hand,z_0 is a pole of the right side of (2.1), its multiplicity is at most τp_1,it follows that $q_1 \le p_1$.there is a contradiction.

(b) If z_0 is a pole of w_1 with multiplicity τ and a pole of some $a_k(z)$ with multiplicity $\tau(a_k,\infty)$ and a zero of some $b_j(z)$ with multiplicity $\tau(b_j,0)$,then

$$\tau(z_0,Q_{p_1}(z,w_1)) \le \tau p_1+\sum_{k=0}^{p_1}\tau(a_k,\infty), \tau(z_0,Q_{q_1}(z,w_1)) \ge \tau q_1-\sum_{j=0}^{q_1}\tau(b_j,0).$$

When $\tau q_1 > \sum_{j=0}^{q_1}\tau(b_j,0)$,we have

$$\tau(z_0,\frac{\Omega_1(z,w_1,w_2)}{(w_2-b)^{\lambda_2}})=\tau(z_0,Q_{p_1})-\tau(z_0,Q_{q_1}) \le \sum_{k=0}^{p_1}\tau(a_k,\infty)+\sum_{j=0}^{q_1}\tau(b_j,0).$$

When $\tau q_1 \leq \sum_{j=0}^{q_1} \tau(b_j,0)$,we have $\tau \leq \frac{1}{q_1}\sum_{j=0}^{q_1} \tau(b_j,0)$. Thus

$$\tau(z_0, \frac{\Omega_1(z,w_1,w_2)}{(w_2-b)^{\lambda_2}}) \leq \sum_{k=0}^{p_1} \tau(a_k,\infty) + \sum_{j=0}^{q_1} \tau(b_j,0). \tag{2.9}$$

Case (III) If z_0 is a pole of w_2 but not any pole of w_1,then

$$\tau(z_0,(\frac{w_2^{(l)}}{w_2-b})^{i_{2l}}) = \tau(z_0,(\frac{(w_2-b)^{(l)}}{w_2-b})^{i_{2l}}) = li_{2l},$$

then

$$\tau(z_0, \frac{\Omega_1(z,w_1,w_2)}{(w_2-b)^{\lambda_2}}) \leq \sum_{(i)} \tau(z_0,a_{(i)}) + u_2. \tag{2.10}$$

Combining the cases (I),(II) and (III),we get the inequality (2.6).

Lemma 3[5] *Let $R(z,w) \equiv P(z,w)/Q(z,w) \equiv \sum_{i=0}^{p} a_i(z)w^i / \sum_{j=0}^{q} b_j w^j$ be an irreducible rational function in w with the meromorphic coefficients $\{a_i(z)\}$ and $\{b_j(z)\}$.If $w(z)$ is a meromorphic function,then $T(r,R(z,w)) = \max\{p,q\}T(r,w)+O\{\sum T(r,a_i)+\sum T(r,b_j)\}$.*

3. Proof of Theorem 1

We assume $q_1 \geq 1$ and $q_2 \geq 1$.Suppose (w_1,w_2) is a meromorphic solution of (1.1),the components w_1 and w_2 are admissible.

$$R_1(z,w_1) = \sum_{i=0}^{p_1} a_i(z)w_1^i / \sum_{j=0}^{q_1} b_j(z)w_1^j = P_{11}(z,w_1) + P_{12}(z,w_1)/\sum_{j=0}^{q_1} b_j(z)w_1^j,$$

$$R_2(z,w_2) = \sum_{i=0}^{p_2} c_i(z)w_2^i / \sum_{j=0}^{q_2} d_j(z)w_2^j = P_{22}(z,w_2) + P_{21}(z,w_2)/\sum_{j=0}^{q_2} d_j(z)w_2^j,$$

where $P_{1k}(z,w_1)(k=1,2)$ is polynomial in w_1 and $\deg P_{12}(z,w_1) < q_1, \deg P_{11}(z,w_1) = \max\{0,p_1-q_1\}$, $P_{2k}(z,w_2)(k=1,2)$ is polynomial in w_2 and $\deg P_{21}(z,w_2) < q_2$, $\deg P_{22}(z,w_2) = \max\{0,p_2-q_2\}$.

Thus,we rewrite (1.1) as follows:

$$\begin{cases} (\frac{\Omega_1}{(w_2-b)^{\lambda_2}} - P_{11}(z,w_1)) \sum_{j=1}^{q_1} b_j(z)w_1^j = P_{12}(z,w_1), \\ (\frac{\Omega_2}{(w_1-a)^{\overline{\lambda}_1}} - P_{22}(z,w_2)) \sum_{j=1}^{q_2} d_j(z)w_2^j = P_{21}(z,w_2). \end{cases} \tag{3.1}$$

Apply Lemma 1,Lemma 2 and Lemma 3 to (3.1), then we get

$$q_1T(r,w_1) + o(T(r,w_1)) \leq \lambda_2 T(r,w_2) + u_2\overline{N}(r,w_2) + S_1(r) \tag{3.2}$$

$$q_2T(r,w_2) + o(T(r,w_2)) \leq \overline{\lambda}_1 T(r,w_1) + \overline{u}_1\overline{N}(r,w_1) + S_1(r) \tag{3.3}$$

Combining (3.2) with (3.3),we have $[q_1+o(1)][q_2+o(1)]\prod_{k=1}^{2}T(r,w_k)\le[\lambda_2+o(1)+u_2\frac{\overline{N}(r,w_2)}{T(r,w_2)}+\frac{S_1(r)}{T(r,w_2)}][\overline{\lambda}_1+o(1)+\overline{u}_1\frac{\overline{N}(r,w_1)}{T(r,w_1)}+\frac{S_1(r)}{T(r,w_1)}]\prod_{k=1}^{2}T(r,w_k)$, as $r\to\infty$,possibly outside a set of finite linear measure. So we get $q_1q_2\le[\lambda_2+u_2(1-\theta(w_2,\infty))][\overline{\lambda}_1+\overline{u}_1(1-\theta(w_1,\infty))]$. This completes the proof of Theorem 1.

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复高阶代数微分方程组的亚纯解

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摘 要: 利用亚纯函数的 Nevanlinna 值分布理论，研究了一类高阶代数微分方程组亚纯允许解的存在性问题，得到了一个主要结果.