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The Problem of Timan on the Precise Order of the Best Approximations of Multivariate Functions *

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Abstract: The problem of Timan on finding a necessary and sufficient condition for

$$\Omega^k(f,\frac{1}{\sigma})_{L_p(R^n)}=O(A_\sigma(f)_{L_p(R^n)}),\quad \sigma\to\infty,$$

is solved. The condition is that

$$\Omega^k(f,\delta)_{L_p(R^n)}=O(\Omega^{k+1}(f,\delta)_{L_p(R^n)}),\quad \delta\to 0.$$

Key words: modulus of continuity; best appraximation; entire function of exponential sphesrical type σ.

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1. Introduction

For $1\le p<\infty$, a function $f=f(x_1,\cdots,x_n)$ of n variables belongs to the space $L_p=L_p(R^n)$, if it has the finite norm

$$\|f\|_p=\{\int_{R^n}|f(x)|^p\mathrm{d}x\}^{\frac{1}{p}},\ 1\le p<\infty;\|f\|_\infty=\mathrm{esssup}_{x\in R^n}|f(x)|,\ p=\infty.$$

Denote by $M_{\sigma p}=M_{\sigma p}(R^n)(1\le p\le\infty)$ the collection of all entire functions of exponential spherical type σ which as functions of a real $x\in R^n$ lie in L_p (see [1]). For any $f\in L_p(R^n)$, the quantity $A_\sigma(f)=\inf_{g_\sigma\in M_{\sigma p}}\|f-g_\sigma\|_{L_p(R^n)}$ is called the best approximation of f by $M_{\sigma p}$. By [1], we know, for any $f\in L_p(R^n)(1<p<\infty)$, there exists $g_\sigma\in M_{\sigma p}$, such that

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$\|f - g_\sigma\| = A_\sigma(f)$. Let $h \in R^n$ be a unit vector, i.e., $|h| = 1$. The modulus of continuity of order k of the function f in the metric of $L_p(R^n)$ along the direction of h is the quantity

$$\omega^k(\delta) = \omega_h^k(f, \delta) = \sup_{|t| \le \delta} \|\Delta_{th}^k f\|_{L_p(R^n)}, \tag{1}$$

where $\Delta_h^k f(x) = \sum_{l=0}^k (-1)^{l+k} C_k^l f(x + lh), (k = 0, 1, \cdots,)$. By [1],

$$\omega(l\delta) \le l\omega(\delta), (\delta > 0, l = 1, 2, \cdots), \omega^k(l\delta) \le l^k \omega^k(\delta), \quad (k, l \in N), \tag{2}$$

$$\omega^k(\delta) \le \omega^k(\delta'), \quad (0 < \delta < \delta').$$

Accordingly,

$$\omega^k(l\delta) \le (l+1)^k \omega^k(\delta), \ \ (l > 0, k \in N), \omega^{k+s}(\delta) \le 2^s \omega^k(\delta). \tag{3}$$

We introduce the quantity

$$\Omega^k(f, \delta) = \Omega^k(f, \delta)_{L_p(R^n)} = \sup_{|h|=1} \omega_h^k(f, \delta)_{L_p(R^n)}, \tag{4}$$

which we will call the modulus of continuity of order k of the function f. Now, suppose that the function f has the derivatives of order ρ. Then it makes sense to speak of the derivative in the direction of any unit vector $h \in R^n$:

$$f_h^{(\rho)} = \sum_{|s|=\rho} f^{(s)} h^s,$$

where $h^s = h_1^{s_1} \cdots h_n^{s_n}, s = (s_1, \cdots, s_n)$. Put $\Omega^k(f^{(\rho)}, \delta) = \sup_{h \in R^n, |h|=1} \omega_h^k(f_h^{(\rho)}, \delta)$, we will call this quantity the modulus of continuity of the derivatives (all of them) of order ρ of the function f. And we have

$$\Omega^k(f^{(\rho)}, l\delta) \le (1+l)^k \Omega^k(f^{(\rho)}, \delta), (l > 0, k \in N).$$

If the derivatives of order ρ of the function f belong to the space $L_p(R^n)$, then

$$\Omega^l(f, \delta) \le \delta^\rho \Omega^k(f^{(\rho)}, \delta), \quad l = k + \rho. \tag{5}$$

For any positive integer k, by [1],

$$A_\sigma(f)_{L_p(R^n)} \le C_k \Omega^k(f, \frac{1}{\sigma})_{L_p(R^n)}, \tag{6}$$

where C_k is independent of f and σ.

In this paper, we concerns conditions on f with which the reverse inequality holds

$$\Omega^k(f, \frac{1}{\sigma})_{L_p(R^n)} \le E A_\sigma(f)_{L_p(R^n)},$$

where E is independent of σ.

For the case $n = 1$, Rathore[2] has the following theorem:

Theorem A *Let m be a positive integer and $f \in L_p(-\infty, +\infty)$, there exists a positive constant G such that*

$$\Omega^m(f, \frac{1}{\sigma})_{L_p(R)} \le GA_\sigma(f)_{L_p(R)}, \quad \sigma > 0, \tag{7}$$

if and only if there exists a positive constant F such that

$$\Omega^m(f, \delta) \le F\Omega^{m+1}(f, \delta), \quad \delta > 0. \tag{8}$$

We generalized this result to $n-$dimensional space, and proved the following result:

Theorem *Let m be a positive integer and $f \in L_p(R^n), (1 < p < \infty)$. There exists a positive constant G such that*

$$\Omega^m(f, \frac{1}{\sigma})_{L_p(R^n)} \le GA_\sigma(f)_{L_p(R^n)}, \quad \sigma > 0, \tag{9}$$

if and only if there exists a positive constant F such that

$$\Omega^m(f, \delta)_{L_p(R^n)} \le F\Omega^{m+1}(f, \delta)_{L_p(R^n)}, \quad (\delta > 0). \tag{10}$$

2. Some basic lemmas

Lemma 1 *For any $f \in L_p(R^n)$ $(1 < p < \infty)$,*

$$\Omega^k(f, \frac{1}{r})_{L_p(R^n)} \le \frac{C_{kn}}{r^k} \sum_{\nu=0}^{r} (\nu + 1)^{k-1} A_\nu(f)_{L_p(R^n)}. \tag{11}$$

Proof If $g_\sigma(f, x) \in M_{\sigma p}$ such that $\|f(x) - g_\sigma(f, x)\|_{L_p(R^n)} = \inf_{g_\sigma \in M_{\sigma p}} \|f - g_\sigma\|_{L_p(R^n)}$, for any positive $r \in N$, we take natural number m such that $2^m \le r < 2^{m+1}$

$$\Omega^k(f, \frac{1}{r}) \le \Omega^k(f - g_{2^{m+1}}, \frac{1}{r}) + \Omega^k(g_{2^{m+1}}, \frac{1}{r}). \tag{12}$$

Moreover,

$$\begin{aligned}\Omega^k(g_{2^{m+1}}, \frac{1}{r})_{L_p(R^n)} &\le \frac{1}{r^k} \sup_{h \in R^n, |h|=1} \{\int_{R^n} | \sum_{|s|=k} g^{(s)}_{2^{m+1}} h^s |^p \mathrm{d}x\}^{\frac{1}{p}} \\ &\le \frac{1}{r^k} \sup_{h \in R^n, |h|=1} \sum_{|s|=k} |h^s| [\{\int_{R^n} |g_1^{(s)}|^p \mathrm{d}x\}^{\frac{1}{p}} + \\ &\sum_{\nu=0}^{m} \{\int_{R^n} |g^{(s)}_{2^{\nu+1}}(f, x) - g^{(s)}_{2^\nu}(f, x)|^p \mathrm{d}x\}^{\frac{1}{p}}],\end{aligned}$$

For every $\nu = 1, 2, \cdots, m$, we have

$$\{\int_{R^n} |g^{(s)}_{2^{\nu+1}}(f,x) - g^{(s)}_{2^\nu}(f,x)|^p dx\}^{\frac{1}{p}} \le (2^{\nu+1})^{|s|}\{\int_{R^n} |g_{2^{\nu+1}}(f,x) - g_{2^\nu}(f,x)|^p dx\}^{\frac{1}{p}} \le 2^{(\nu+1)k+1} A_{2^\nu}(f)$$

and

$$\{\int_{R^n} |g^{(s)}_1(f,x)|^p dx\}^{\frac{1}{p}} = \{\int_{R^n} |g^{(s)}_1(f,x) - g^{(s)}_0(f,x)|^p dx\}^{\frac{1}{p}} \le 2A_0(f).$$

It follows that $\Omega^k(g_{2^{m+1}}, \frac{1}{r})_{L_p(R^n)} \le c(k,n)\frac{2}{r^k}[A_0(f) + \sum_{\nu=0}^m 2^{(\nu+1)k} A_{2^\nu}(f)]$. Taking account also of the fact that

$$2^{(\nu+1)k} A_{2^\nu}(f) \le 2^{2k} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{k-1} A_\mu(f), \tag{13}$$

we obtain the estimate

$$\Omega^k(g_{2^{m+1}}, \frac{1}{r}) \le c(k,n)\frac{2^{2k+1}}{r^k}[A_0(f) + A_1(f) + \sum_{\nu=1}^{m} \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{k-1} A_\mu(f)]$$

$$\le \frac{c_{kn}}{r^k} \sum_{\nu=0}^{2^m} (\nu+1)^{k-1} A_\nu(f). \tag{14}$$

By (4) and (13), we have

$$\Omega^k(f - g_{2^{m+1}}, \frac{1}{r}) \le \frac{c_k}{r^k} \sum_{\mu=0}^{2^m} (\mu+1)^{k-1} A_\mu(f). \tag{15}$$

Combining (14), (15) and (12), we arrive at the inequality (11).

Lemma 2 *If*

$$|t|^k \int_{|t|\le|u|\le 1} \frac{\Omega^k(f,|u|)}{|u|^{k+n}} du = O(\Omega^k(f,|t|)), \quad |t| \to 0, \tag{16}$$

then

$$\Omega^k(|t|) = \Omega^k(f,|t|) = O[\frac{1}{v^k |lnv|} \Omega^k(v|t|)] \tag{17}$$

uniformly with respect to all positive $v \le \frac{1}{2}$ *and* $0 < |t| \le \frac{1}{2}$.

Proof For any positive $v \le \frac{1}{2}$ and $0 < |t| \le \frac{1}{2}$, take positive integer m, r, such that

$\frac{1}{r+1} < |t| \le \frac{1}{r}, \frac{1}{m+1} < v \le \frac{1}{m}$,

$$\begin{aligned}|t|^k \int_{|t|\le|u|\le 1} \frac{\Omega^k(f,|u|)}{|u|^{k+n}}du &\ge \frac{1}{(r+1)^k}\int_{\frac{1}{r}\le|u|\le 1}\frac{\Omega^k(f,|u|)}{|u|^{k+n}}du \\ &= \frac{1}{(r+1)^k}\sum_{\nu=1}^{r-1}\int_{\frac{1}{\nu+1}\le|u|\le\frac{1}{\nu}}\frac{\Omega^k(f,|u|)}{|u|^{k+n}}du \\ &\ge \frac{1}{(r+1)^k}\sum_{\nu=1}^{r-1}\Omega^k(f,\frac{1}{\nu+1})\int_{\frac{1}{\nu+1}\le|u|\le\frac{1}{\nu}}\frac{1}{|u|^{k+n}}du \\ &\ge \frac{a_{kn}}{(r+1)^k}\sum_{\nu=1}^{r-1}\nu^{k-1}\Omega^k(f,\frac{1}{\nu}).\end{aligned}$$

It follows that when (16) is satisfied

$$\frac{1}{(m+1)^k(r+1)^k}\sum_{\nu=1}^{mr}\nu^{k-1}\Omega^k(\frac{1}{\nu}) \le b_{kn}\Omega^k(f,\frac{1}{mr}). \tag{18}$$

Moreover,

$$\begin{aligned}\frac{1}{(r+1)^k}\sum_{\nu=1}^{mr}\nu^{k-1}\Omega^k(\frac{1}{\nu}) &= \frac{1}{(r+1)^k}\sum_{j=1}^{m}\sum_{\nu=(j-1)r+1}^{jr}\nu^{k-1}\Omega^k(\frac{1}{\nu}) \\ &\ge \frac{1}{(r+1)^k}\sum_{j=1}^{m}\Omega^k(\frac{1}{jr})\sum_{\nu=(j-1)r+1}^{jr}\nu^{k-1} \\ &\ge \alpha_k\sum_{j=1}^{m}\Omega^k(\frac{1}{jr})j^{k-1}.\end{aligned}$$

For any positive integer $j \ge 1$, we have the inequality :

$$\Omega^k(\frac{1}{r}) \le j^k\Omega^k(\frac{1}{jr}).$$

Thus,

$$\frac{1}{(r+1)^k}\sum_{\nu=1}^{mr}\nu^{k-1}\Omega^k(\frac{1}{\nu}) \ge \alpha_k\Omega^k(\frac{1}{r})\sum_{j=1}^{m}\frac{1}{j},$$

i.e.,

$$\Omega^k(\frac{1}{r}) = O(\{\frac{1}{(r+1)^k\ln m}\sum_{\nu=1}^{mr}\nu^{k-1}\Omega^k(\frac{1}{\nu})\}).$$

Combining the last relation with the inequality (18), we obtain

$$\Omega^k(\frac{1}{r}) = O(\frac{m^k}{\ln m}\Omega^k(\frac{1}{mr})),$$

or

$$\Omega^k(|t|)=O(\frac{1}{v^k|\ln v|}\Omega^k(v|t|)).$$

Lemma 3 *If condition (16) holds, then*

$$\frac{1}{r^k}\sum_{\nu=0}^{r}(\nu+1)^{k-1}A_\nu(f)_{L_p(R^n)}=O(A_r(f)_{L_p(R^n)}),\quad (r\to\infty). \tag{19}$$

Proof If condition (16) holds, by inequality (6),

$$\frac{1}{r^k}\sum_{\nu=0}^{r}(\nu+1)^{k-1}A_\nu(f)_{L_p(R^n)}\le\frac{C_{kn}}{r^k}\sum_{\nu=0}^{r}(\nu+1)^{k-1}\Omega^k(f,\frac{1}{\nu+1})_{L_p(R^n)}$$

$$=O(\frac{1}{(r+2)^k}\int_{\frac{1}{r+2}\le|u|\le1}\frac{\Omega^k(f,|u|)}{|u|^{k+n}}du)\le\beta_k\Omega^k(f,\frac{1}{r}),(r\ge1). \tag{20}$$

Besides, for any integer $m\ge1$ we have

$$\Omega^k(f,\frac{1}{mR})\le\frac{C_k}{m^kr^k}\sum_{\nu=0}^{mr}(\nu+1)^{k-1}A_\nu(f)$$

Consequently,

$$\sum_{\nu=r+1}^{mr}(\nu+1)^{k-1}A_\nu(f)_{L_p(R^n)}\ge\frac{m^kr^k}{C_k}\Omega^k(f,\frac{1}{mr})_{L_p(R^n)}-\beta_kr^k\Omega^k(f,\frac{1}{r})_{L_p(R^n)},$$

i.e.,

$$A_r(f)_{L_p(R^n)}\sum_{\nu=r+1}^{mr}(\nu+1)^{k-1}\ge\frac{m^kr^k}{C_k}\Omega^k(f,\frac{1}{mr})_{L_p(R^n)}-\beta_kr^k\Omega^k(f,\frac{1}{r})_{L_p(R^n)}$$

or

$$A_r(f)_{L_p(R^n)}\ge\frac{m^kr^k}{C_k(m+1)^k(r+1)^k}\Omega^k(f,\frac{1}{mr})_{L_p(R^n)}-\beta_k\frac{r^k}{(m+1)^k(r+1)^k}\Omega^k(f,\frac{1}{r})_{L_p(R^n)}$$
$$\ge\frac{1}{C_k4^k}\Omega^k(f,\frac{1}{mr})_{L_p(R^n)}-\frac{\beta_k}{m^k}\Omega^k(f,\frac{1}{r})_{L_p(R^n)}.$$

By Lemma 2,

$$\Omega^k(f,\frac{1}{r})_{L_p(R^n)}=O(\frac{m^k}{\ln m}\Omega^k(f,\frac{1}{mr}))_{L_p(R^n)}.$$

It follows that m may be choosen in such a way that ,

$$\frac{\beta_k}{m^k}\Omega^k(f,\frac{1}{r})_{L_p(R^n)}\le\frac{1}{C_k8^k}\Omega^k(f,\frac{1}{mr})_{L_p(R^n)}.$$

Thus,

$$A_r(f) \geq \frac{1}{C_k 8^k}\Omega^k(f,\frac{1}{mr})_{L_p(R^n)} \geq \frac{1}{C_k 8^k m^k}\Omega^k(f,\frac{1}{r})_{L_p(R^n)},$$

or

$$\Omega^k(f,\frac{1}{r})_{L_p(R^n)} \leq M_k A_r(f). \tag{21}$$

Combining (20), (21), we obtain (19).

Lemma 4 *If condition (16) holds, then*

$$\Omega^k(f,\frac{1}{\sigma})_{L_p(R^n)} \leq E A_\sigma(f). \tag{22}$$

Proof For any $\sigma > 0$, take a positive integer N, such that $N \leq \sigma < N+1$, by Lemma 3, we have

$$\begin{aligned}\Omega^k(f,\frac{1}{\sigma})_{L_p(R^n)} &\leq \Omega^k(f,\frac{1}{N})_{L_p(R^n)} \leq \frac{B_k}{N^k}\sum_{\nu=0}^{N}(\nu+1)^{k-1}A_\nu(f)_{L_p(R^n)} \\ &\leq \frac{B_k}{N^k}\sum_{\nu=0}^{N+1}(\nu+1)^{k-1}A_\nu(f)_{L_p(R^n)} \leq 2^k B_k C_k A_{N+1}(f)_{L_p(R^n)} \\ &\leq 2^k B_k C_k A_\sigma(f)_{L_p(R^n)}.\end{aligned}$$

Put $E = 2^k B_k C_k$, then (22) holds.

Lemma 5 *If condition (19) holds, then (16) follows.*

Proof For any $|t| > 0$, take a positive integer r, such that $\frac{1}{r+1} < |t| \leq \frac{1}{r}$

$$\begin{aligned}|t|^k\int_{|t|\leq|u|\leq 1}\frac{\Omega^k(f,|u|)}{|u|^{k+n}}du &\leq \frac{1}{r^k}\int_{\frac{1}{r+1}\leq|u|\leq 1}\frac{\Omega^k(f,|u|)}{|u|^{k+n}}du \\ &= \frac{1}{r^k}\sum_{\nu=1}^{r}\int_{\frac{1}{\nu+1}\leq|u|\leq\frac{1}{\nu}}\frac{\Omega^k(f,|u|)}{|u|^{k+n}}du \\ &\leq \frac{1}{r^k}\sum_{\nu=1}^{r}\Omega^k(f,\frac{1}{\nu})\int_{\frac{1}{\nu+1}\leq|u|\leq\frac{1}{\nu}}\frac{1}{|u|^{k+n}}du \\ &\leq \frac{B_k}{r^k}\sum_{\nu=1}^{r}\nu^{k-1}\Omega^k(f,\frac{1}{\nu}).\end{aligned}$$

By condition (19) and Lemma 1, we have

$$\Omega^k(f,\frac{1}{r}) \leq \frac{C_k}{r^k}\sum_{\nu=0}^{r}(\nu+1)^{k-1}A_\nu(f) \leq C_k B_k A_r(f) = C_k^* A_r(f),\ (r\to\infty).$$

Thus,

$$\begin{aligned}&|t|^k\int_{|t|\leq|u|\leq 1}\frac{\Omega^k(f,|u|)}{|u|^{k+n}}du \leq \frac{B_k}{r^k}\sum_{\nu=0}^{r}\nu^{k-1}\Omega^k(f,\frac{1}{\nu}) \leq \frac{C_k^* B_k}{r^k}\sum_{\nu=0}^{r}\nu^{k-1}A_\nu(f) \\ &= O(A_r(f)) = O(\Omega^k(f,\frac{1}{r+1})) = O(\Omega^k(f,|t|)),\ (|t|\to 0).\end{aligned}$$

3. The proof of theorem

Proof If (9) holds, using (6) for $k=m+1$, (10) follows with $F=GC_{m+1}$. Conversely, if (10) holds, using the inequality

$$\Omega^m(f,|u|)\le(1+\frac{|u|}{|t|})^m\Omega^m(f,|t|),\quad (|t|\le|u|),$$

we get

$$|t|^{m+1}\int_{|t|\le|u|\le1}\frac{\Omega^{m+1}(f,|u|)}{|u|^{m+1+n}}du\le2|t|^{m+1}\int_{|t|\le|u|\le1}\frac{\Omega^m(f,|u|)}{|u|^{m+1+n}}du$$
$$\le2|t|^{m+1}\Omega^m(f,|t|)\int_{|t|\le|u|\le1}[\frac{(1+\frac{|u|}{|t|})^m}{|u|^{m+1+n}}]du$$
$$\le C_n2^{m+1}\Omega^m(f,|t|)\le C_n2^{m+1}F\Omega^{m+1}(f,|t|).$$

Thus, (16) holds with $k=m+1$, so that for some constant E independent of σ, we have

$$\Omega^m(f,\frac{1}{\sigma})\le F\Omega^{m+1}(f,\frac{1}{\sigma})\le FEA_\sigma(f)$$

and (9) follows with $G=FE$. We complete the proof.

By Lemma 1–Lemma 5 and Theorem, we have

Corollary *If $k<m$, for the following statements there holds (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (v)* *(i)* $\Omega^k(f,\frac{1}{\sigma})=O(A_\sigma(f))(\sigma\to\infty)$.

(ii) $\Omega^k(f,\delta)=O(\Omega^m(f,\delta))(\delta\to0)$.

(iii) $|t|^m\int_{|t|\le|u|\le1}\frac{\Omega^m(f,|u|)}{|u|^{m+n}}du=O(\Omega^m(f,|t|))(|t|\to0)$.

(iv) $\frac{1}{N^m}\sum_{\nu=0}^{N}(\nu+1)^{m-1}A_\nu(f)=O(A_N(f))(N\to\infty)$.

(v) $\Omega^m(f,\frac{1}{\sigma})=O(A_\sigma(f))(\sigma\to\infty)$.

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关于多元函数最佳逼近精确阶的 Timan 问题

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摘 要: 关于找一个充分必要条件使 $\Omega^k(f,\frac{1}{\sigma})_{L_p(R^n)}=O(A_\sigma(f)_{L_p(R^n)})(\sigma\to\infty)$ 成立的 Timan 问题被解决. 这个条件是 $\Omega^k(f,\delta)_{L_p(R^n)}=O(C_2\Omega^{k+1}(f,\delta)_{L_p(R^n)}),\delta\to0$.