Journal of Mathematical Research & Exposition
Vol.21, No.2, 191–196, May, 2001

On Non-Decomposable Hermitian Forms over $Z[\sqrt{-5}]$ *

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Abstract: In this paper, we discuss the non-decomposability of lattices over $Z[\sqrt{-5}]$. All lattices of rank 2 with discriminant 2 are found and the lattices of rank $n \geq 3$ with discriminant 2 are constructed.

Key words: non-decomposable form; lattice; class.

Classification: AMS(1991) 11E39/CLC O156.5

Document code: A **Article ID:** 1000-341X(2001)02-0191-06

1. Introduction

Let $F = Q(\sqrt{-m})$ ($m > 0$ and square-free) be an imaginary quadratic field, $\mathbf{o}$ the ring of integers of F, (V, H) a Hermitian space of dimension n with a postive definite Hermitian form H. A lattice L in V is called integral if $H(x, y) \in \mathbf{o}$ for all $x, y \in L$. In this respect, H is also called the Hermitian form on L. In this paper, all lattices (if not specified) will be integral with respect to H.

Let L be a lattice in a Hermitian space (V, H). It is well-known that there exists a base $\{x_1, x_2, \ldots, x_n\}$ and ideals $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ in F, suth that $L = \mathbf{a}_1x_1 + \mathbf{a}_2x_2 + \ldots + \mathbf{a}_nx_n$, and $\{x_i\}$ and $\{\mathbf{a}_i\}$ can be chosen in a way suth that $\mathbf{a}_1 = \mathbf{a}_2 = \cdots = \mathbf{a}_{n-1} = \mathbf{o}$ ([1, 81:3] and [1,81:5]).

Definition 1.1 *Let $L = \mathbf{a}_1x_1 + \mathbf{a}_2x_2 + \cdots + \mathbf{a}_nx_n$ be a lattice in (V, H). The ideal $\mathbf{d}(L) = \det H(x_i, x_j)\prod_1^n \mathbf{a}_i\bar{\mathbf{a}}_j$ is called the discriminant of L. If $\mathbf{d}(L) = d\mathbf{o}$ $(d \in N)$, we simply write $d(L) = d$.*

It is clear that if L is integral, then $\mathbf{d}(L) \subseteq \mathbf{o}$. It can be shown that $\mathbf{d}(L)$ is independent of the choice of $\{x_i\}$ and $\{\mathbf{a}_i\}$ ([2]).

Definition 1.2 *Let H be a positive definite Hermitian form on L. Then H, or alternatively L, is called decomposable if there exist two non-trivial positive semi-definite Hermitian forms H_1 and H_2 on L such that*

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(1) $H(x,y) = H_1(x,y) + H_2(x,y)$;

(2) $H_i(x,y) \in \mathbf{o}$, $i = 1,2$,

for all $x, y \in L$. *Otherwise* H, *also* L, *is called non-decomposable.*

It is easy to prove

Proposition 1.1 *Let* $L = \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \ldots + \mathbf{a}_n x_n$ *be an integral lattice in* (V, H), $A = (H(x_i, x_j))$ *the matrix of* L. *Then* H *is decomposable on* L *if and only if there exist two non-trivial positive semi-definite Hermitian matrices* $B = (b_{ij})$ *and* $C = (c_{ij})$ *such that (1)* $A = B + C$; *(2)* $b_{ij}\mathbf{a}_i\bar{\mathbf{a}}_j \subseteq \mathbf{o}$, $c_{ij}\mathbf{a}_i\bar{\mathbf{a}}_j \subseteq \mathbf{o}$, $\quad i,j = 1,2,\ldots,n$.

For non-decomposability concerning unimodular lattices Zhu gave a complete resolution ([3]). He also discussed the decomposability of positive definite Hermitian forms over Gaussian domain ([4]). The main purpose of this paper is to discuss the decomposability of positive definite Hermitian forms over $Z[\sqrt{-5}\,]$ with discriminant $d = 2$. We shall prove the following theorems:

Theorem 1 *There are exactly nine classes of positive definite Hermitian forms of rank 2 with discriminant 2 over* $Z[\sqrt{-5}\,]$, *and the representatives are:* $[1,\ 0,\ 1]^{\mathbf{p}}$, $[2,\ 1,\ 1]^{\mathbf{p}}$, $[2,\ \omega,\ 3]^{\mathbf{p}}$, $[3,\ \omega,\ 2]^{\mathbf{p}}$, $[5,\ 3+\omega,\ 3]^{\mathbf{p}}$, $[5,\ 2(1+\omega),\ 5]^{\mathbf{p}}$, $[1,\ 0,\ 2]$, $[2,\ 1+\omega,\ 4]$, $[7,\ 3(1+\omega),\ 8]$. *All are decompasable except two non-free lattices* $[3,\ \omega,\ 2]^{\mathbf{p}}$ *and* $[5,\ 3+\omega,\ 3]^{\mathbf{p}}$. *Where* $\omega = \sqrt{-5}$, $\mathbf{p} = (2,\ 1+\sqrt{-5}\,)$, *and* $[a,\ b,\ c]^{\mathbf{p}}$ *denotes the lattice* $L = \mathbf{o}x_1 + \mathbf{p}x_2$ *with the matrix* $\left(\begin{smallmatrix} a & b \\ \bar{b} & c \end{smallmatrix}\right)$.

Theorem 2 *For every* $n > 3$ *there exist* n*-ary non-decomposable positive definite Hermitian free-lattices over* $Z[\sqrt{-5}\,]$ *with discriminant* $d = 2$. *There are no such forms with the desired properties for* $n = 1$ *or* 2.

2. The lattices of rank 2 with discriminant 2

Definition 2.1 *Let* L, K *be two lattices in a Hermitian space* (V, H), $\mathbf{p}$ *be a prime ideal such that* $\bar{\mathbf{p}} = \mathbf{p}$. L *is said to be* $\mathbf{p}$*-close to* K *if the invariant factors of* L *in* K *are* $\mathbf{p}, \mathbf{o}, \mathbf{o}, \cdots, \mathbf{o}$ *or* $\mathbf{o}, \mathbf{o} \cdots \mathbf{o}, \mathbf{p}^{-1}$.

It is clear that L is $\mathbf{p}$-close to K if and only if K is $\mathbf{p}$-close to L.

Proposition 2.1 *Let* L *be a lattice in* (V, H), $y \in L \backslash \mathbf{p}L$ *such that* $\mathbf{p}^r \parallel H(y, L)$. *Then we have that* $K = L(y) = \{\, x \in L \mid H(x,y) \equiv 0 \pmod{\mathbf{p}^{r+1}} \,\}$ *is* $\mathbf{p}$*-close to* L.

Proof Let $L = \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \ldots + \mathbf{a}_n x_n$. Without loss of generality, we can assume that $\mathbf{a}_i$ are integral and $(\mathbf{a}_i, \mathbf{p}) = 1$ $(i = 1,2,\ldots,\ n)$, $\mathbf{p}^{r_i} \parallel H(x,\ y)$ $\quad (r = r_1 \le r_i \le r_{i+1},\ i = 1,\ 2,\ \ldots,\ n-1)$. Take an element $\alpha \in \mathbf{a}_1 \setminus \mathbf{p}$, then $\alpha\mathbf{a}_i \subseteq \mathbf{a}_1$ and $\mathbf{p}^{r_i} \parallel H(\alpha^{-1}x_i,\ y)$. Therefore we can assume that $\mathbf{a}_i \subseteq \mathbf{a}_1 \subseteq \mathbf{o} \quad (i = 2,\ 3,\ \ldots,\ n)$.

By the strong Approximation Theorem ([1,21:8]), there exist $\alpha_i \in F \quad (i = 1,2,\cdots,n)$, such that

$$\begin{cases} |H(x_i,y) + \bar{\alpha}_i H(x_1,y)|_p \le \varepsilon, & i = 2,\ 3,\ \ldots,\ n. \\ |\alpha_i|_q \le 1, & \forall q \ne p. \end{cases}$$

For every positive real number ε. And for sufficiently small positive number ε, $\alpha_i \in \mathbf{o}$.

We have

$$L = \mathfrak{a}_1 x_1 + \mathfrak{a}_2(x_2 + \alpha_2 x_1) + \ldots + \mathfrak{a}_n(x_n + \alpha_n x_1).$$

Hence $L' = \mathfrak{a}_1\mathfrak{p}x_1 + \mathfrak{a}_2(x_2+\alpha_2 x_1)+\ldots+\mathfrak{a}_n(x_n+\alpha_n x_1)$ is $\mathfrak{p}$-close to L. We have $L' \subseteq L(y)$ and $L(y) \neq L$ since $\mathfrak{a}_1 x_1 \not\subseteq L(y)$. This implies $L' = L$. Therefore $L(y)$ is $\mathfrak{p}$-close to L.

Proposition 2.2 *Let L be a unimodular lattice, K a sublattice of L. If K is $\mathfrak{p}$-close to L, then there exists an element $y \in L \setminus \mathfrak{p}L$ such that*

$$K = L(y) = \{x \in L \mid H(x, y) \equiv 0 \pmod{\mathfrak{p}}\}.$$

And y is independ of the choice in $[y] \bmod \mathfrak{p}L$.

Proof There exists a base $\{x_i\}$ of V and ideals $\{\mathfrak{a}_i\}$ such that

$$L = \mathfrak{a}_1 x_1 + \mathfrak{a}_2 x_2 + \ldots + \mathfrak{a}_n x_n; \quad k = \mathfrak{a}_1\mathfrak{p}x_1 + \mathfrak{a}_2 x_2 + \ldots + \mathfrak{a}_n x_n, \quad (\mathfrak{a}_1, \mathfrak{p}) = 1.$$

Then $L^{\#} = \mathfrak{a}_1^{-1}x_1^{\#} + \mathfrak{a}_2^{-1}x_2^{\#} + \ldots + \mathfrak{a}_n^{-1}x_n^{\#}$. Since L is a unimodular lattice, $L = L^{\#}$. Choose an element $\beta \in \mathfrak{a}_1^{-1}\backslash\mathfrak{p}$, then $\beta x_1^{\#} \in L$ and $\mathfrak{p} \mid H(\beta x_1^{\#}, L)$.

Therefore $K = L(\beta x_1^{\#}) = \{x \in L \mid H(x, \beta x_1^{\#}) \equiv 0 \pmod{\mathfrak{p}}\}$.

Next, if $y \equiv \beta x_1^{\#} \pmod{\mathfrak{p}L}$, then

$$H(x, y) \equiv 0 \pmod{\mathfrak{p}} \text{ if and only if } H(x, \beta x_1^{\#}) \equiv 0 \pmod{\mathfrak{p}}.$$

This implies the result. □

Let $F = Q(\sqrt{-5})$, $\mathfrak{o} = Z[\sqrt{-5}]$, $\omega = \sqrt{-5}$, $\mathfrak{p} = (2, 1+\omega)$.

Proposition 2.3 *Let (V, H) be a Hermitian space of dimension 2, K a lattice in V with discriminant $d(K) = 2$. Then there exists a unimodular lattice L in V such that $K \subseteq L$ and K is $\mathfrak{p}$-close to L.*

Proof Let $K = \mathfrak{o}x_1 + \mathfrak{a}x_2$. Since the class number of F is 2, $\mathfrak{a}$ can be chosen such that either $\mathfrak{a} = \mathfrak{o}$ or $\mathfrak{a} = \mathfrak{p}^{-1}$.

Suppose $K = \mathfrak{o}x_1 + \mathfrak{p}^{-1}x_2$, then $2 \mid H(x_2, x_2)$ and $\det(H(x_i, x_j)) = 4$.

(a). If $2 \mid H(x_1, x_1)$, then $\mathfrak{p}^2 \mid H(x_1, x_2)$. Then $L = \mathfrak{p}^{-1}x_1 + \mathfrak{p}^{-1}x_2$ a desired.

(b). If $4 \mid H(x_2, x_2)$, then $\mathfrak{p}^2 \| H(x_1, x_2)$. Then $L = \mathfrak{o}x_1 + \mathfrak{o}(2^{-1}x_2)$ a desired.

(c). If $2 \nmid H(x_1, x_1)$, and $2 \| H(x_2, x_2)$, Let $y_1 = \frac{1+\omega}{2}x_2, y_2 = x_2$. Then $K = \mathfrak{o}y_1 + \mathfrak{p}^{-1}y_2$, with $2 \mid H(y_1, y_1)$. Therefore $L = \mathfrak{p}^{-1}y_1 + \mathfrak{p}^{-1}y_2$ a desired in(a). □

Proposition 2.4[2] *There are exactly six classes of positive definite Hermitian unimodular lattice of rank 2. The representative forms are*

$[1, 0, 1]$, $[2, \omega, 3]$, $[5, 2(1+\omega), 3]$, $[1, 0, 2]^{\mathfrak{p}^{-1}}$, $[2, 1+\omega, 4]^{\mathfrak{p}^{-1}}$, $[2, 1+\bar{\omega}, 4]^{\mathfrak{p}^{-1}}$.

Proof of Theorem 1 Let K be a lattice of rank 2 with discriminant 2. There is a unimodular lattice L of rank 2 and an element $y \in L$ such that $K = L(y)$ and K is $\mathfrak{p}$-close to L. And such y is independent of the choice in $[y] \bmod \mathfrak{p}L$ by Proposition 2.1 and 2.2. Hence all classes of rank 2 with discriminant 2 can be found from the unimodular lattices in Proposition 2.3.

Let L be a unimodular lattice of rank 2. Then there is a base x_1, x_2 of V such that either $L = \mathbf{o}x_1 + \mathbf{o}x_2$ with $\det A = 1$ or $L = \mathbf{o}x_1 + \mathbf{p}^{-1}x_2$ with $\det A = 2$. For any $y \in L \subseteq \mathbf{p}L$, we have

$$y \equiv \begin{cases} x_1, \\ x_2, \\ x_1 + x_2, \end{cases} \pmod{\mathbf{p}L}$$

while $L = \mathbf{o}x_1 + \mathbf{o}x_2$, or

$$y \equiv \begin{cases} x_1, \\ \frac{1+\omega}{2}x_2, \\ x_1 + \frac{1+\omega}{2}x_2, \end{cases} \pmod{\mathbf{p}L}$$

where $L = \mathbf{o}x_1 + \mathbf{p}^{-1}x_2$.

By constructing $\mathbf{p}$-close lattices for all lattices in Proposition 2.3 and for all y listed above, we get the following nine classes:

$[1, 0, 1]^{\mathbf{P}}$, $[2, 1, 1]^{\mathbf{P}}$, $[2, \omega, 3]^{\mathbf{P}}$, $[3, \omega, 2]^{\mathbf{P}}$, $[5, 3+\omega, 3]^{\mathbf{P}}$, $[5, 2(1+\omega), 5]^{\mathbf{P}}$, $[1, 0, 2]$, $[2, 1+\omega, 4]$, $[7, 3(1+\omega), 8]$.

It can be shown that they are not pair-wisely equivalent from by counting the elements representing 1, 2, 3 and their coefficients.

Now we show that $[3, \omega, 2]^{\mathbf{P}}$ is non-decomposable.

Suppose that there exist two non-trivial positive semi-definite Hermitian matrices $A = (a_{ij})_2$ and $B = (b_{ij})_2$ such that

(1). $\begin{pmatrix} 3 & \omega \\ \bar{\omega} & 2 \end{pmatrix} = A + B$;

(2). $a_{11}, b_{11} \in Z$, $a_{22}, b_{22} \in \frac{1}{2}Z$, $a_{12}, b_{12} \in \mathbf{p}^{-1}$. We have $a_{22} = 0, 1, \frac{1}{2}, \frac{3}{2}$ or 2 since $a_{22} \in \frac{1}{2}Z$ and $0 \le a_{22} \le 2$.

(a). If $a_{22} = 0$, then $a_{12} = a_{21} = 0$. Hence $B = \begin{pmatrix} 3 & \omega \\ \bar{\omega} & 2 \end{pmatrix}$ since $\det B > 0$. This implies $A = 0$, a contradiction.

(b). If $a_{22} \ge 1$, then $b_{22} \le 1$. Hence $\mathrm{N}(b_{12}) \le 3$ since $\det B \ge 0$. But $b_{12} \in \mathbf{p}^{-1}$, $2\mathrm{N}(b_{12}) \in Z$ and $2\mathrm{N}(b_{12} \le 6$. Threrfore $b_{12} = 0, \pm 1, \pm\frac{1\pm\omega}{2}$, which is impossible since each case implies a contradiction.

Therefore we have only trivial decomposition. this proves the non-decomposability for $[3, \omega, 2]^{\mathbf{P}}$.

By the same method we can show that $[5, 3+\omega, 3]^{\mathbf{P}}$ is also non-decomposable.

For the other lattices, it is easy to verify that they are all decomposable by exhibiting the decomposition for each one. For example, we have $[2, \omega, 3]^{\mathbf{P}} = [2, 2\omega, 12]^{\mathbf{P}^{-1}} = [1, 1+\omega, 6]^{\mathbf{P}^{-1}} + [1, -1+\omega, 6]^{\mathbf{P}^{-1}}$ and $[7, 3(1+\omega), 8] = [2, 1+\omega, 3] + [5, 2(1+\omega), 5]$.

The same method applies for the other lattices.

Since the only two non-decomposable lattices are not free, all free lattices listed in Theorem 1 are decomposable. This complete the proof. □

3. Non-decomposable lattices of rank $n \ge 3$ with discriminant 2

Proposition 3.1 *The free-lattices represented by*

$$H_1 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & \omega \\ 0 & \bar{\omega} & 4 \end{pmatrix}, \qquad \det H_1 = 2,$$

$$H_2 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1+\omega \\ 0 & 0 & 1+\bar{\omega} & 5 \end{pmatrix}, \qquad \det H_2 = 2$$

are non-decomposable.

The proof follows directly from [4, Lemma 4].

Now

$$\Lambda = \begin{pmatrix} 3 & \omega \\ \bar{\omega} & 4 \end{pmatrix}; \qquad \widetilde{\Lambda} = \begin{pmatrix} 4 & \omega \\ \bar{\omega} & 3 \end{pmatrix};$$

$$H_0 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & \omega \\ 0 & \bar{\omega} & 4 \end{pmatrix}; \qquad \widetilde{H}_0 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1+\omega \\ 0 & 0 & 1+\bar{\omega} & 5 \end{pmatrix};$$

$$H_g = \begin{pmatrix} & & 0 & 0 \\ & H_{g-1} & \vdots & \vdots \\ & & 0 & 0 \\ & & 1 & 0 \\ 0 \cdots 0 & 1 & & \\ 0 \cdots 0 & 0 & \multicolumn{2}{c}{\Lambda} \end{pmatrix}; \qquad \widetilde{H}_g = \begin{pmatrix} \multicolumn{2}{c}{\widetilde{\Lambda}} & 0 & 0 \cdots 0 \\ & & 1 & 0 \cdots 0 \\ 0 & 1 & & \\ \vdots & \vdots & \multicolumn{2}{c}{\widetilde{H}_{g-1}} \\ 0 & 0 & & \end{pmatrix}.$$

We have

Proposition 3.2 *The free-lattices represented by H_g and $\widetilde{H}_g$ are non-decomposable of ranks $n = 2g+3$ and $n = 2g+4$ respectively with discriminant $d = 2$ for $g = 0, 1, 2, \ldots$.*

Proof We give the proof for H_g only, since it also applies to $\widetilde{H}_g$.

Let $A_0 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $A_g = \begin{pmatrix} H_{g-1} & 1 \\ 1 & 3 \end{pmatrix}$ $(g \geq 1)$. We have $\det A_i = 3$ and $\det H_i = 2$ $(i \geq 0)$. It follows that H_g are positive definite Hermititian matrices for all non-negative integars g.

Now, we prove the non-decomposability of H_g by induction.

First of all, H_0 is non-decomposable by proposition 3.1. Assume H_g is non-decomposable for $g \geq 0$, and consider H_{g+1}.

Suppose we have a decomposition as $H_{g+1} = D_1 + D_2$, where D_1 and D_2 are positive semi-definite Hermitian matrices. Write

$$H_{g+1} = \begin{pmatrix} L_1 & * \\ * & \Lambda_1 \end{pmatrix} + \begin{pmatrix} L_2 & * \\ * & \Lambda_2 \end{pmatrix},$$

where L_i are $(2g+3)$-th matrices and Λ_i are 2×2 matrices $(i = 1, 2)$. This gives a decomposition of H_g: $H_g = L_1 + L_2$. Since H_g is non-decomposable by assumption,

$L_1 = 0$ or $L_2 = 0$. Suppose $L_1 = 0$, then $D_1 = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_1 \end{pmatrix}$ since D_1 is a positive semi-definite Hermitian matrix. Therefore

$$H_{g+1} = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_1 \end{pmatrix} + \begin{pmatrix} H_g & 1 \\ & 1\ \Lambda_2 \end{pmatrix}.$$

Let $\Lambda_2 = \begin{pmatrix} a & \beta \\ \bar{\beta} & b \end{pmatrix}$. By computing the cofactors we have $ab - \beta\bar{\beta} \geq 0, 2(ab - \beta\bar{\beta}) - 3b \geq 0, 2a - 3 \geq 0$. Hence $a \geq 2$ and $b(2a-3) - 2\beta\bar{\beta} \geq 0$.

(1). If $a = 2$, then $N(\beta) = 0$ or 1 .

(a). If $N(\beta) = 0$, then $\Lambda_1 = \left(\begin{smallmatrix} 1 & \omega \\ \bar{\omega} & * \end{smallmatrix}\right)$ is not positive semi-definite since $b \leq 4$.

(b). If $N(\beta) = 1$, then $\Lambda_1 = \left(\begin{smallmatrix} 1 & \pm 1+\omega \\ \pm 1+\bar{\omega} & * \end{smallmatrix}\right)$ is also not positive semi-definite.

(2). If $a = 3$, then $\Lambda_1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & * \end{smallmatrix}\right)$ and $\Lambda_2 = \left(\begin{smallmatrix} 3 & \omega \\ \bar{\omega} & b \end{smallmatrix}\right)$. Since $b \leq 4$, and $\det \Lambda_2 \geq 0$, we have $b = 4$. Hance $\Lambda_1 = 0$ which means that we have only trivial decomposition. This proves the non-decomposability of H_g. □

Proof of theorem 2 Proposition 3.1 and 3.2 gives the non-decomposable free-lattices with discriminant 2 for $n \geq 3$. Theorem 1 shows there are only two non-decomposable lattices (both not free) of rank 2 with discriminant 2, and the proof for dcomposability of lattices of rank 1 with discriminant 2 is trivial. This completes the proof. □

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$Z[\sqrt{-5}]$ 上不可分的 Hermite 型

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摘 要: 本文讨论了 $Z[\sqrt{-5}]$ 上不可分的正定 Hermite 型的构作. 给出了所有秩为 2 判别式等于 2 的不可分的正定 Hermite 型. 当秩 $n \geq 3$ 时，证明了存在 $Z[\sqrt{-5}]$ 上判别式等于 2 的不可分的正定 Hermite 型，并给出了它们的明显结构.