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Exponential Attractors for the Ginzburg-Landau-BBM Equations *

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Abstract: In this paper, the existence of the exponential attractors for the Ginzburg-Landau-BBM equations with periodic initial and boundary conditions are obtained by using the squeezing property and the operator decomposition method.

Key words: exponential attractor; Ginzburg-Landau-BBM equations; squeezing property.

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1. Introduction

The infinite dimensional dynamical systems which are defined by nonlinear evolution equations have a lot of properties similar to what the differential dynamical systems have, such as attractors and inertial manifolds. However, because of spectral barriers and spectral gap conditions, for many dissipative evolution equations such as the 2D Navier-Stokes equations, the existence of an inertial manifold is still a question. The exponential attractors are an important feature for describing the long-time behavior of solutions of the nonlinear evolution equations. The major difference between inertial manifolds and exponential attractors is that the latter do not assume any global slaving of small scales. Therefore, the exponential attractors can deal with cases where an exponential convergence is not restricted within a manifold structure. Many researches have been done on exponential attractors such as Eden A. et al[1] in dissipative evolution equations, Babin A. et al[2] in reaction-diffusion systems in an unbounded domain, Dai Z.D. et al[3] in nonlinear wave equations, Gao H.J.[4] in generalized Ginzburg-Landau equation and so on.

In this paper, we consider the following system of the Ginzburg-Landau-BBM equations

$$\varepsilon_t = \mu\varepsilon + (\alpha_1 + i\alpha_2)\varepsilon_{xx} - (\beta_1 + i\beta_2)|\varepsilon|^2\varepsilon + i\delta n\varepsilon, \tag{1.1}$$

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$$n_t + f(n)_x + \gamma n - \nu n_{xx} - n_{xxt} + |\varepsilon|^2_x = g(x), \tag{1.2}$$

with periodic initial value

$$\varepsilon(x+L,t) = \varepsilon(x,t), \quad n(x+L,t) = n(x,t), \tag{1.3}$$

$$\varepsilon(x,0) = \varepsilon_0(x), \quad n(x,0) = n_0(x). \tag{1.4}$$

and prove the existence of the exponential attractors of problem (1.1)-(1.4). The existence of global attractors and its estimates of the upper bounds of Hausdorff and fractal dimensions have been studied in [5].

2. Exponential attractors and squeezing property

Let $\mathcal{H}$ be a separable Hilbert space and B be a compact subset of $\mathcal{H}$, $\{S(t)\}_{t\geq 0}$ be a nonlinear continuous semigroup that leaves the set B invariant and set

$$\mathcal{A} = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} S(t)B},$$

that is for $\{S(t)\}_{t\geq 0}$ on B, $\mathcal{A}$ is the global attractor.

Definition 1 *A set $\mathcal{M}$ is called an exponential attractors for $(\{S(t)\}_{t\geq 0}, B)$ if*

(i) $\mathcal{A} \subseteq \mathcal{M} \subseteq B$;

(ii) $S(t)\mathcal{M} \subseteq \mathcal{M}$, for every $t \geq 0$;

(iii) There exist constants c_1 and c_2 such that

$$\mathrm{dist}_{\mathcal{H}}(S(t)B, \mathcal{M}) \leq c_1 \exp(-c_2 t), \quad \forall t \geq 0;$$

(iv) $\mathcal{M}$ has finite fractal dimension.

Definition 2 *A continuous semigroup $\{S(t)\}_{t\geq 0}$ is said to satisfy the squeezing property on B if there is $t_* > 0$ such that $S_* = S(t_*)$ satisfies: there exists an orthogonal projection $P = P(N_0)$ of rank N_0 such that for every u and v in B, if*

$$\|P(S_*u - S_*v)\|_{\mathcal{H}} \leq \|(I-P)(S_*u - S_*v)\|_{\mathcal{H}},$$

then

$$\|S_*u - S_*v\|_{\mathcal{H}} \leq \frac{1}{8}\|u - v\|_{\mathcal{H}}.$$

Lemma 1[1] *If $(\{S(t)\}_{t\geq 0}, B)$ satisfies the squeezing property on B and if $S_* = S(t_*)$ is Lipschitz on B with Lipschitz constant L, then there exists an exponential attractor $\mathcal{M}$ for $(\{S(t)\}_{t\geq 0}, B)$ such that*

$$d_F(\mathcal{M}) \leq N_0 \max\{1, \frac{\ln(16L+1)}{\ln 2}\},$$

and

$$\mathrm{dist}_{\mathcal{H}}(S(t)B, \mathcal{M}) \leq c_1 \exp(-\frac{c_2}{t_*}t),$$

where c_1, c_2 are the constants independent of u_0 and t.

It is clear that, we need only to find t_*, N_0 and to prove the squeezing property in $H^2 \times H^2$ for problem (1.1)-(1.4).

3. Exponential attractors for problem (1.1)-(1.4)

Let $\Omega = [0, L]$, $H^2 = H^2_{per}(\Omega) = \{u : D^\sigma u \in L^2(\Omega), \sigma = 0, 1, 2\}$. By [5], we know that the operator semigroup $S(t)$ is completely continuous in $H^2 \times H^2$ and there exists a bounded absorbing set B_0 of $S(t)$ where for $t \geq t_0$

$$B_0 = \{(\varepsilon(t), n(t)) \in H^2 \times H^2 : \|\varepsilon(t)\|^2_{H^2} + \|n(t)\|^2_{H^2} \leq \rho\},$$

where t_0, ρ are constants depending on $\|\varepsilon_0(x)\|_{H^2}$, $\|n_0(x)\|_{H^2}$ and the coefficients of (1.1)-(1.2).

Let

$$B = \overline{\bigcup_{t \geq t_0} S(t) B_0},$$

then B is a compact invariant subset in $H^2 \times H^2$. $\{\lambda_j\}_{j \in N}$ denote the eigenvalues of $-\partial_{xx}$ with periodic boundary condition, and $\{\omega_j\}_{j \in N}$ are the eigenfunctions with respect to $\{\lambda_j\}_{j \in N}$. Let

$$X_N = \mathrm{Span}\{\omega_1, \cdots, \omega_N\},$$

and $P_N : L^2(\Omega) \to X_N$ be the orthogonal projection onto X_N and $Q_N = I - P_N$.

We will show the squeezing property by these orthogonal projections.

Assume that $(\varepsilon_1, n_1), (\varepsilon_2, n_2)$ are two solutions of problem (1.1)-(1.4) with initial value $(\varepsilon_{10}, n_{10})$ and $(\varepsilon_{20}, n_{20})$ respectively. By [5], we have

$$(\varepsilon_1, n_1), \quad (\varepsilon_2, n_2) \in L^\infty(0, T; H^2 \times H^2).$$

Set $U = \varepsilon_1 - \varepsilon_2, V = n_1 - n_2$ which satisfy

$$U_t = \mu U + (\alpha_1 + i\alpha_2) U_{xx} - (\beta_1 + i\beta_2)(|\varepsilon_1|^2 U + |\varepsilon_2|^2 U + \varepsilon_1 \varepsilon_2 \overline{U}) + i\delta(n_1 U + \varepsilon_2 V), \quad (3.1)$$

$$V_t + (f'(\xi)V)_x + \gamma V - \nu V_{xx} - V_{xxt} + (\varepsilon_1 \overline{U} + \overline{\varepsilon}_2 U)_x = 0. \quad (3.2)$$

Taking the real part of the inner product of (3.1) with $U, -U_{xx}$ and U_{xxxx} respectively, we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U\|^2 - \mu\|U\|^2 + \alpha_1\|U_x\|^2 + \\ &\quad \mathrm{Re}(\beta_1 + i\beta_2)(|\varepsilon_1|^2 U + |\varepsilon_2|^2 U + \varepsilon_1\varepsilon_2\overline{U}, U) + \mathrm{Im}\delta(n_1 U + \varepsilon_2 V, U) = 0, \quad (3.3) \\ &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U_x\|^2 - \mu\|U_x\|^2 + \alpha_1\|U_{xx}\|^2 \\ &\quad = \mathrm{Re}(\beta_1 + i\beta_2)(|\varepsilon_1|^2 U + |\varepsilon_2|^2 U + \varepsilon_1\varepsilon_2\overline{U}, U_{xx}) + \mathrm{Im}\delta(n_1 U + \varepsilon_2 V, U_{xx}), \quad (3.4) \\ &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U_{xx}\|^2 - \mu\|U_{xx}\|^2 + \alpha_1\|U_{xxx}\|^2 \\ &\quad = \mathrm{Re}(\beta_1 + i\beta_2)((|\varepsilon_1|^2 U + |\varepsilon_2|^2 U + \varepsilon_1\varepsilon_2\overline{U})_x, U_{xxx}) + \\ &\quad \mathrm{Im}\delta((n_1 U + \varepsilon_2 V)_x, U_{xxx}), \quad (3.5) \end{aligned}$$

where

$$|\mathrm{Re}(\beta_1+i\beta_2)(|\varepsilon_1|^2U+|\varepsilon_2|^2U+\varepsilon_1\varepsilon_2\overline{U},U_{xx})|\le\frac{\alpha_1}{4}\|U_{xx}\|^2+C(\alpha_1)\|U\|^2,$$

$$|\mathrm{Im}\delta(n_1U+\varepsilon_2V,U_{xx})|\le\frac{\alpha_1}{4}\|U_{xx}\|^2+C(\alpha_1)(\|U\|^2+\|V\|^2),$$

$$|\mathrm{Re}(\beta_1+i\beta_2)((|\varepsilon_1|^2U+|\varepsilon_2|^2U+\varepsilon_1\varepsilon_2\overline{U})_x,U_{xxx})|\le\frac{\alpha_1}{8}\|U_{xxx}\|^2+C(\alpha_1)(\|U\|^2+\|U_x\|^2),$$

$$|\mathrm{Im}\delta((n_1U+\varepsilon_2V)_x,U_{xxx})|\le\frac{\alpha_1}{8}\|U_{xxx}\|^2+C(\alpha_1)(\|U\|^2+\|U_x\|^2+\|V\|^2+\|V_x\|^2).$$

Note

$$\mu\|U_{xx}\|^2\le\mu\|U_x\|\,\|U_{xxx}\|\le\frac{\alpha_1}{4}\|U_{xxx}\|^2+C(\alpha_1,\mu)\|U_x\|^2.$$

Combining the above inequalities, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|U\|^2+\|U_x\|^2+\|U_{xx}\|^2)+2\alpha_1\|U_x\|^2+\alpha_1\|U_{xx}\|^2+\frac{\alpha_1}{2}\|U_{xxx}\|^2\\&\quad\le C(\alpha_1,\mu)(\|U\|^2+\|U_x\|^2+\|V\|^2+\|V_x\|^2).\end{aligned}\tag{3.6}$$

On the other hand, taking the inner product of (3.2) with V and $-V_{xx}$ respectively, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|V\|^2+\|V_x\|^2)+\gamma\|V\|^2+\nu\|V_x\|^2=(f'(\xi)V,V_x)+(\varepsilon_1\overline{U}+\overline{\varepsilon}_2U,V_x),\tag{3.7}$$

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|V_x\|^2+\|V_{xx}\|^2)+\gamma\|V_x\|^2+\nu\|V_{xx}\|^2\\&\quad=((f'(\xi)V)_x,V_{xx})+((\varepsilon_1\overline{U}+\overline{\varepsilon}_2U)_x,V_{xx}).\end{aligned}\tag{3.8}$$

Similarly

$$|((f'(\xi)V)_x,V_{xx})|\le\tfrac{\nu}{4}\|V_{xx}\|^2+C(\nu)(\|V\|^2+\|V_x\|^2),$$
$$|((\varepsilon_1\overline{U}+\overline{\varepsilon}_2U)_x,V_{xx})|\le\tfrac{\nu}{4}\|V_{xx}\|^2+C(\nu)(\|U\|^2+\|U_x\|^2).$$

Thus, from (3.7)(3.8) one gets

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|V\|^2+\|V_x\|^2+\|V_{xx}\|^2)+2\gamma\|V\|^2+2(\nu+\gamma)\|V_x\|^2+\nu\|V_{xx}\|^2\\&\quad\le C(\nu)(\|U\|^2+\|V\|^2+\|U_x\|^2+\|V_x\|^2).\end{aligned}\tag{3.9}$$

Combining (3.6) and (3.9), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|U\|_{H^2}^2+\|V\|_{H^2}^2)+c_0(\|U\|_{H^2}^2+\|V\|_{H^2}^2)\le C(\alpha_1,\nu,\mu)(\|U\|_{H^1}^2+\|V\|_{H^1}^2).\tag{3.10}$$

By Gronwall's inequality

$$\|U(t)\|_{H^2}^2+\|V(t)\|_{H^2}^2\le e^{kt}(\|U(0)\|_{H^2}^2+\|V(0)\|_{H^2}^2).$$

This gives the Lipschitz continuity with $L = \mathrm{Lip}_{H^2\times H^2}(S(t)) \le e^{kt}$.

Decompose U, V as

$$U = P_N U + Q_N U, \qquad V = P_N V + Q_N V$$

where P_N, Q_N are orthogonal operators and P is N-dimension. Using $\|u_x\| \le \frac{1}{2}\|u\|^2 + \frac{1}{2}\|u_{xx}\|^2$, from (3.10) we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|Q_N U\|_{H^2}^2 + \|Q_N V\|_{H^2}^2) + c_0(\|Q_N U\|_{H^2}^2 + \|Q_N V\|_{H^2}^2)\\ &\le C(\alpha_1,\nu,\mu)\lambda_{N+1}^{-1}(\|Q_N U_{xx}\|^2 + \|Q_N V_{xx}\|^2)\\ &\le C(\alpha_1,\nu,\mu)\lambda_{N+1}^{-1}(\|Q_N U\|_{H^2}^2 + \|Q_N V\|_{H^2}^2)\\ &\le C(\alpha_1,\nu,\mu)\lambda_{N+1}^{-1}e^{kt}(\|U(0)\|_{H^2}^2 + \|V(0)\|_{H^2}^2).\end{aligned}$$

Integrate above inequality, obtains

$$\begin{aligned}\|Q_N U\|_{H^2}^2 + \|Q_N V\|_{H^2}^2 \le & e^{-c_0 t}(\|Q_N U(0)\|_{H^2}^2 + \|Q_N V(0)\|_{H^2}^2) +\\ & \lambda_{N+1}^{-1} k e^{kt}(\|U(0)\|_{H^2}^2 + \|V(0)\|_{H^2}^2).\end{aligned}$$

If we choose N_0 large enough such that

$$\lambda_{N_0+1} \ge 256 k e^{kt_*},$$

where t_* satisfies

$$e^{kt_*} \le \frac{1}{4} \times \left(\frac{1}{8}\right)^2,$$

then from

$$\|P_{N_0} u(t_*)\|_{H^2}^2 \le \|Q_{N_0} u(t_*)\|_{H^2}^2$$

we deduce that

$$\begin{aligned}\|U(t_*)\|_{H^2}^2 + \|V(t_*)\|_{H^2}^2 &\le 2(\|Q_{N_0} U(t_*)\|_{H^2}^2 + \|Q_{N_0} V(t_*)\|_{H^2}^2)\\ &\le \left(\frac{1}{8}\right)^2 (\|U(0)\|_{H^2}^2 + \|V(0)\|_{H^2}^2).\end{aligned}$$

Then, we get the squeezing property in $H^2 \times H^2$. By Lemma 1, the following theorem holds.

Theorem 1 *Suppose that $\varepsilon_0(x), n_0(x) \in H^2, g(x) \in H^1$ and $f(n) \in C^2$, then the problem (1.1)-(1.4) admits a compact exponential attractor in $H^2 \times H^2$.*

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Ginzburg-Landau-BBM 方程的指数吸引子

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摘 要: 本文利用挤压性质和算子分解方法证明了 Ginzburg-Landau-BBM 耦合方程组周期初边值问题指数吸引子的存在性.