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Combinatorial and Statistical Applications of Generalized Stirling Numbers *

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Abstract: Here illustrated are certain combinatorial and statistical applications of the generalized Stirling numbers with integer parameters.

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1. Introduction

In 1984, Charalambides and Koutras[2] obtained some combinatorial and statistical applications of the Gould-Hopper numbers $G(n,k;r,s)$. One interesting application is the occupancy distribution which is useful in biology and reliability.

Recently, L.C.Hsu and P.J-S.Shiue[9] defined a kind of generalized Stirling numbers $S^1(n,k) \equiv S(n,k;\alpha,\beta,\gamma)$ and established several properties analogous to that of the classical one. More properties were obtained by R.B.Corcino[6]. The Gould-Hopper numbers can be considered as Stirlingtype numbers, in the sense that it can be expressed in terms of $S(n,k;\alpha,\beta,\gamma)$ with $\alpha=1,\beta=r,\gamma=s$. In particular, we have

$$G(n,k;r,s) \equiv r^k S(n,k;1,r,s).$$

This motivates the authors to investigate the analogous combinatorial and statistical applications of the generalized Stirling numbers.

In this paper, we will discuss some combinatorial and statistical applications of the generalized Stirling numbers $S^1(n,k)$ parallel to that obtained by Charalambides and Koutras for the Gould-Hopper numbers.

2. Preliminaries

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Let us define the generalized Stirling numbers $S(n, k; \alpha, \beta, \gamma)$ based on the definition of L.C. Hsu and P.J-S. Shiue[9] as follows

$$(t|\alpha)_n = \sum_{k=0}^{n} S(n, k; \alpha, \beta, \gamma)(t - \gamma|\beta)_n,$$

where α, β, γ may be real or complex not all zero and $(t|\alpha)_n$ denotes the generalized factorial of the form

$$(t|\alpha)_n = \prod_{j=0}^{n-1}(t - j\alpha), \quad n \geq 1$$

with $(t|\alpha)_0 = 1$. In particular, $(t|1)_n = (t)_n$ and $(t)_0 = 1$.

We need the following results from [6]:

[R1] $\beta^k k! S(n, k; \alpha, \beta, \gamma) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}(\beta i + \gamma|\alpha)_n$;

[R2] $\beta^k k!(-1)^{n+k} S(n, k; \alpha, -\beta, -\gamma) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}(\beta i + \gamma| - \alpha)_n$ (This is a direct consequence of [R1]);

[R3] $(\beta k + \gamma|\alpha)_n = \sum_{i=0}^{n}(k)_i \beta^k S(n, i; \alpha, \beta, \gamma)$;

[R4] $\sigma_k(x) = \sum_{n \geq k} S^1(n-1, k-1)\frac{1}{(x|\alpha)_n} = \frac{1}{(x-\gamma|\beta)_k}$.

3. Combinatorial interpretations

Let α, β, γ be nonnegative integers with α dividing both β and γ, that is, $\alpha|\beta, \alpha|\gamma$.

Distributing balls into cells.

Consider $k+1$ distinct cells, the first k of which each has β distinct compartments, and the last cell with γ distinct compartments. The compartments in each cell are given cyclic ordered numbering and

(A1) the capacity of each compartment is limited to one ball.

Suppose we distribute n distinct balls into the $k+1$ cells, one ball at a time such that

(B1) each successive α available compartments in a cell can only have the leading compartment getting the ball

(C1) the first k cells are nonempty.

Illustration of (B1):

Suppose the first ball lands in compartment 3 of cell 2. The compartment numbered $4, 5, 6, \cdots, \alpha, \alpha+1, \alpha+2$ will be closed. And suppose the second ball lands in compartment $\beta-2$ also of cell 2. Then compartments numbered $\beta-1, \beta, 1, 2, \alpha+3, \alpha+4, \alpha+5, \cdots, 2\alpha-3$ of cell 2 will be closed.

Problem How many ways can we distribute n distinct balls into $k+1$ distinct cells one ball at a time under restrictions

(i) (A1) and (B1)?

(ii) (A1), (B1), and (C1)?

Solution of (i) Let Ω be the set of all possible ways of distributing the n balls under restrictions (A1) and (B1). Then

$$\begin{aligned}|\Omega| &= (\beta k + \gamma)(\beta k + \gamma - \alpha)(\beta k + \gamma - 2\alpha)\cdots(\beta k + \gamma - (n-1)\alpha) \\ &= (\beta k + \gamma|\alpha)_n.\end{aligned}$$

Solution of (ii) Define property $P_i (i = 1, 2, 3, \cdots, k)$:

$$P_i \text{ holds} \Leftrightarrow i\text{th cell is empty}$$

Let $\omega(s)$ be the number of outcomes in Ω satisfying at least s properties, that is, at least s out of k cells are empty. Then

$$\omega(s) = \binom{k}{s}(\beta(k-s) + \gamma|\alpha)_n.$$

By Principle of Inclusion and Exclusion, the number of outcomes in Ω satisfying none of the k properties, that is, none of the first k cells is empty, is

$$\begin{aligned} \sum_{s=0}^{k}(-1)^s\omega(s) &= \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}(\beta i + \gamma|\alpha)_n \\ &= \beta^k k! S(n, k; \alpha, \beta, \gamma) \text{ for [R1]}. \quad \square \end{aligned}$$

These results are embodied in the following proposition.

Proposition 1 *Let Ω be the sample space associated with above problem satisfying A(1) and (B1). Then*

$$|\Omega| = (\beta k + \gamma|\alpha)_n$$

and the number of outcomes in Ω satisfying (C1) is

$$E_\Omega(0) = \beta^k k! S(n, k; \alpha, \beta, \gamma)$$

with $\alpha > 0, \beta, \gamma \geq 0, \alpha|\beta, \alpha|\gamma$.

Suppose we modify conditions (A1) and (B1) as follows

(A1)* capacity of each compartment is unlimited.

(B1)* after a ball lands in a compartment, this compartment splits into $\alpha + 1$ compartments, each of unlimited capacity.

Then using similar argument we used to solve the preceding problem, we can easily prove the following proposition with the help of [R2].

Proposition 2 *Let Ω^* be the new sample space satisfying (A1)* and (B1)*. Then*

$$|\Omega^*| = (\beta k + \gamma| - \alpha)_n$$

and the number of outcomes in Ω^* satisfying (C1) is

$$E^*_\Omega(0) = \beta^k k! (-1)^{n+k} S(n, k; \alpha, \beta, \gamma)$$

with $\alpha, \beta, \gamma \geq 0$.

Remark Proposition 1 and 2 subsume the results of Charalambides and Koutras[2] with $\alpha = 1$. Clearly, the number of surjections $k!S(n, k)$ and the combinatorial interpretations mentioned in [5] for $S(n,k)$ can be deduced from Proposition 2 by putting $\alpha=\gamma=0, \beta=1$ with cells assumed to be indistinguishable and knowing that

$$(-1)^{n+k}S(n,k;0,-1,0)=S(n,k;0,1,0)=S(n,k).$$

Drawing balls from an urn.

Consider an urn containing $k+1$ types of balls such that there are β distinct balls for each of the first k types and γ distinct balls of the last type. The balls in each type are given cyclic ordered numbering. Suppose that n balls are drawn from the urn.

(A2) one after the other without replacement

(B2) each draw automatically includes the successive available $\alpha-1$ balls of the same type

(C2) the first k types of balls are represented in the sample drawn.

We then have the following proposition.

Proposition 3 *Let Γ be the sample space of the preceding experiment satisfying (A2) and (B2). Then*

$$|\Gamma|=|\Omega|=(\beta k+\gamma|\alpha)_n$$

and the number of outcomes in Γ satisfying (C2) is

$$E_\Gamma(0)=\beta^k k! S(n,k;\alpha,\beta,\gamma)$$

with $\alpha>0, \beta,\gamma\geq 0, \alpha|\beta, \alpha|\gamma$.

Suppose we modify the conditions (A2) and (B2) as follows

(A2)* one after the other with replacement

(B2)* each draw automatically puts back α distinct balls of same type drawn.

The following proposition follows immediately.

Proposition 4 *Let Γ^* be the new sample space satisfying (A2)* and (B2)*. Then*

$$|\Gamma^*|=|\Omega^*|=(\beta k+\gamma|-\alpha)_n$$

and the number of outcomes in Γ^* satisfying (C2) is

$$E^*_\Gamma(0)=\beta^k k!(-1)^{n+k}S(n,k;\alpha,\beta,\gamma)$$

with $\alpha>0, \beta,\gamma\geq 0$.

4. Probability distributions and unbiased estimators

Throughout this section, Ω and Ω^* refer to the sample spaces in Section 3.

Occupancy distribution.

In the experiment with Ω as the sample space, let X be the number of occupied cells among the first k cells. Then we have

Proposition 5 *The probability function of X is*

$$P(i|n,k;\alpha,\beta,\gamma)=\frac{(k)_i\beta^k S(n,i;\alpha,\beta,\gamma)}{(\beta k+\gamma|\alpha)_n},$$

where $i = 0, 1, 2, \cdots, \min\{n, k\}$.

Proof Using the combinatorial interpretation given in Proposition 1 and (R3) in Section 2 with $(k)_i = 0$ for $i = k+1, k+2, \cdots$, we have

$$\sum_{i=0}^{k} P(i|n, k; \alpha, \beta, \gamma) = 1. \qquad \square$$

The content and proof of the following proposition is analogous to that of Proposition 5.

Proposition 6 *The probability function of X^* (=number of occupied cells among the first k cells in the sample space Ω^*) is*

$$P^*(i|n, k; \alpha, \beta, \gamma) = \frac{(k)_i \beta^i |S(n, i; \alpha, -\beta, -\gamma)|}{(\beta k + \gamma | -\alpha)_n},$$

where $i = 0, 1, 2, \cdots, \min\{n, k\}$.

The next proposition gives us the unbiased estimator of the number k of cells when $k \le n$.

Proposition 7 *An unbiased estimator of the number k of cells ($k \le n$) associated with the random sampling defined on Ω is*

$$\mu(i, n) := \frac{S(n+1, i; \alpha, \beta, \gamma)}{\beta S(n, i; \alpha, \beta, \gamma)} + \frac{n\alpha - \gamma}{\beta},$$

where $i = 0, 1, 2, \cdots, k$.

Proof It suffices to show

$$E[\mu(i, n)] = k. \qquad ((*))$$

We know that

$$E[\mu(i, n)] = \sum_{i=0}^{n+1} \mu(i, n) P(i|n, k; \alpha, \beta, \gamma).$$

Using [R3] in Section 2 and Proposition 5, we easily obtain $(*)$. $\square$

Proposition 8 *An unbiased estimator of the number k of cells ($k \le n$) associated with the random sampling defined on Ω^* is*

$$\mu^*(i, n) := \frac{|S(n+1, i; \alpha, -\beta, -\gamma)|}{\beta |S(n, i; \alpha, -\beta, -\gamma)|} - \frac{n\alpha + \gamma}{\beta},$$

where $i = 0, 1, 2, \cdots, k$.

The proof of Proposition 8 is analogous to that of Proposition 7.

Sequential occupancy.

In the experiment with Ω as the sample space, suppose that distinct balls are sequentially distributed into the $k+1$ cells until a predetermined number i of cells among the first k cells is occupied by at least one ball. Let Y be the number of balls required. Then we have

Proposition 9 *The probability function of Y is*

$$Q(n|i,k;\alpha,\beta,\gamma)=\frac{(k)_i\beta^i S(n-1,i-1;\alpha,\beta,\gamma)}{(\beta k+\gamma|\alpha)_n},$$

where $n=i,i+1,\cdots,\frac{\beta}{\alpha}(i-1)+\frac{\gamma}{\alpha}+1$.

Proof By multiplicative rule, we have

$$\begin{aligned} Q(n|i,k;\alpha,\beta,\gamma) &= P(i-1|n-1,k;\alpha,\beta,\gamma)\frac{\beta(k-(i-1))}{\beta k+\gamma-\alpha(n-1)} \\ &= \frac{(k)_i\beta^i S(n-1,i-1;\alpha,\beta,\gamma)}{(\beta k+\gamma|\alpha)_n}. \end{aligned}$$

Using [R4] with x replaced by $\beta k+\gamma$, we can easily prove that $Q(n|i,k;\alpha,\beta,\gamma)$ is a probability function. □

With respect to the experiment with Ω^* as the sample space, we have

Proposition 10 *The probability function of Y is*

$$Q^*(n|i,k;\alpha,\beta,\gamma)=\frac{(k)_i\beta^i|S(n-1,i-1;\alpha,-\beta,-\gamma)|}{(\beta k+\gamma|-\alpha)_n},$$

where $n=i,i+1,i+2,\cdots$.

Coupons collector problem.

Consider an urn containing $k+\nu$ distinct types of coupons, each type with β distinct coupons. Let the coupons in each type be given cyclic ordered numbering. Suppose that n coupons are drawn from the urn one coupon at a time without replacement such that each draw automatically includes the successive available $\alpha-1$ coupons. Let Z be the number of types among k specified kinds appearing in the sample. Using Proposition 3 with $\gamma=\beta\nu$, we have the following probability function of Z which can be proved using the argument as that in Proposition 5.

Proposition 11 *The probability function of Z is*

$$p(i|n,k;\alpha,\beta,\nu)=\frac{(k)_i\beta^i S(n,i;\alpha,\beta,\beta\nu)}{(\beta k+\beta\nu|\alpha)_n},$$

where $i=0,1,2,\cdots,\min\{n,k\}$.

Suppose that the coupons are drawn one after the other without replacement and that each draw automatically includes the successive available $\alpha-1$ coupons until a predetermined number i of types among the k specified types apper in the sample. Let Z^* be the number of coupons required. As in Proposition 9 putting $\gamma=\beta\nu$, we can easily prove the following proposition.

Proposition 12 *The probability function of Z^* is*

$$p^*(n|i,k;\alpha,\beta,\nu) = \frac{(k)_i\beta^i S(n-1,i-1;\alpha,\beta,\beta\nu)}{(\beta k+\beta\nu|\alpha)_n},$$

where $n = i, i+1, \cdots, \frac{\beta}{\alpha}(i-1)+\frac{\beta\nu}{\alpha}+1$.

The foregoing results are in fact a kind of generalization of those obtained by Charalambides and Koutras in [2].

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广义 Stirling 数在组合与统计方面的应用

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摘 要: 本文给出了广义 Stirling 数 $S(n,k;\alpha,\beta,\gamma)$ 的组合解释以及对统计学方面的某些应用，包括 Charalambides-Koutras 等人的结果为特例.