Journal of Mathematical Research & Exposition
Vol.21, No.3, 344-348, August, 2001

On the Joint Distribution of the Future Infimum and Its Location for a Transient Bessel Process *

SHAO Xian-xi[1], *YIN Chuan-cun*[2]

(1. Dept. of International Business and Economics, Qingdao University, Shandong 266071, China;
2. Dept. of Math., Qufu Normal University, Shandong 273165, China)

Abstract: For a transient Bessel process X let $I(t) = inf_{s \geq t} X(s)$ and $\xi(t) = inf\{u \geq t : X(u) = I(t)\}$. In this note we compute the joint distribution of $I(t), \xi(t)$ and X_t.

Key words: transient Bessel process; future infimum; location of the future infimum; joint distribution

Classification: AMS(1991) 60J65/CLC O211.62

Document code: A **Article ID:** 1000-341X(2001)03-0344-05

1. Introduction

Let $X = \{X(t), t \geq 0\}$ be a Bessel process of index $\nu = \frac{d}{2} - 1$, i.e., a one-dimensional diffusion on $[0, \infty)$. It is known that X is transient($i.e. lim_{t \to \infty} X(t) = \infty$ almost surely) if and only if $d > 2$. When d is an integer,X can be realized as the radial part of an R^d-valued Brownian motion. Unless stated otherwise, we shall assume $d > 2$ throughout the note.

For $t > 0$,define

$$I(t) = \inf_{s \geq t} X(s)$$

and

$$\xi(t) = \inf\{u \geq t : X(u) = I(t)\}.$$

In other words, for any given $t \geq 0, I(t)$ denotes the future infimum process associated with X, and $\xi(t)$ denotes the (almost surely unique) location of the infimum of X over $[t, \infty)$.

There have been several contributions on future infima of some transient progress. For example,Khoshnevisan et. al.(1994)studied the rate of escape of $I(t)$,see also Khoshnevisan(1995) and Hu & Shi(1995). There also have many papers dealing with random

*Received date: 1998-07-21
Foundation item: Supported by National Natural Science Foundation of China (19801020)
Biography: SHAO Xian-xi, associate professor.
E-mail: f.t.Dept@qdu.edu.cn

time $\xi(t)$. For example, Williams(1970,1974) proved a path decomposition theorem at $\xi(t)$ respectively in case of Brownian motion and linear diffusions; Shi,Z.(1996) established a limsup behavior of $\xi(t)$, see also the references therein.

The main aim of this paper is to derive the joint distribution of $I(t), \xi(t)$ and $X(t)$. The P_0-density of $I(1)$ is well known, we refer to Khoshnevisan et. al.(1994).

2. Distribution of $(I(t), \xi(t), X(t))$

Let $I(t), \xi(t)$ and $X(t)$ be as above,and denote by P_x the probability measure associated with X when started from x. The main result of this paper is the following theorem.

Theorem 1 *For a fixed $t > 0$,we have for $x \geq y, z \geq y, s \geq t$,*

$$P_x(\xi(t) \in \mathrm{d}s, I(t) \in \mathrm{d}y, X(t) \in \mathrm{d}z) = \frac{d-2}{y} P_x(X(t) \in \mathrm{d}z) P_z(\tau_y \in \mathrm{d}s - t)\mathrm{d}y,$$

where(see Kent(1978))

$$P_x(X(t) \in \mathrm{d}y) = t^{-1}x^{-\nu} I_\nu(\frac{xy}{t}) \exp(-\frac{1}{2t}(x^2 + y^2))y^{\nu+1},$$

and I_ν is the usual modified Bessel function,and

$$\begin{aligned} &P_z(\tau_y \in \mathrm{d}s - t)/\mathrm{d}s \\ &= -\frac{1}{\pi}(\frac{y}{z})^\nu \int_0^\infty \frac{\lambda e^{-\frac{1}{2}\lambda^2(s-t)}(J_\nu(\lambda z)N_\nu(\lambda y) - J_\nu(\lambda y)N_\nu(\lambda z))}{J_\nu^2(\lambda y) + N_\nu^2(\lambda y)} \mathrm{d}\lambda, \end{aligned}$$

where J_ν and N_ν are the first kind and second kind Bessel functins of order ν.

The proof of the theorem is based on the following lemma.

Lemma *For any $x \geq y$ and $s > 0$, we have*

$$P_x(\xi(0) \in \mathrm{d}s, I(0) \in \mathrm{d}y) = \frac{d-2}{y} P_x(\tau_y \in \mathrm{d}s)\mathrm{d}y,$$

where

$$P_x(\tau_y \in \mathrm{d}s)/\mathrm{d}s = -\frac{1}{\pi}(\frac{y}{x})^\nu \int_0^\infty \frac{\lambda e^{-\frac{1}{2}\lambda^2 s}(J_\nu(\lambda x)N_\nu(\lambda y) - J_\nu(\lambda y)N_\nu(\lambda x))}{J_\nu^2(\lambda y) + N_\nu^2(\lambda y)} \mathrm{d}\lambda,$$

where J_ν and N_ν are the first kind and second kind Bessel functions of order ν.

Proof For any $w \leq y \leq x, s > 0$,by the Markov property

$$\begin{aligned} P_x(\tau_y < s, I(0) > w) &= P_x(\tau_y < s, \tau_w = \infty) \\ &= P_x(P_{X(\tau_y)}(\tau_w = \infty), \tau_y < s) \\ &= P_y(\tau_w = \infty) P_x(\tau_y < s) \\ &= (1 - (\frac{w}{y})^{2\nu}) P_x(\tau_y < s). \end{aligned}$$

Thus

$$P_x(\tau_y \in ds, I(0) \in dw) = \frac{2\nu}{y}(\frac{w}{y})^{2\nu-1} P_x(\tau_y \in ds)dw,$$

in particular,

$$P_x(\tau_y \in ds, I(0) \in dy) = \frac{2\nu}{y} P_x(\tau_y \in ds)dy.$$

Therefore

$$P_x(\xi(0) \in ds, I(0) \in dy) = P_x(\tau_y \in ds, I(0) \in dy) = \frac{d-2}{y} P_x(\tau_y \in ds)dy.$$

Proof of the theorem For $s \geq t, x \geq y$ and $z \geq y$, by the Markov property

$$\begin{aligned}
&P_x(\xi(t) > s, I(t) > y, X(t) \in dz) \\
&= P_x(\inf_{t \leq u \leq s} X(u) > \inf_{u \geq s} X(u), \inf_{u \geq t} X(u) > y, X(t) \in dz) \\
&= P_x(P_{X(t)}(\inf_{t \leq u \leq s-t} X(u) > \inf_{u \geq s-t} X(u), \inf_{u \geq 0} X(u) > y), X(t) \in dz) \\
&= P_x(P_{X(t)}(\xi(0) > s-t, \inf_{u \geq 0} X(u) > y), X(t) \in dz) \\
&= P_x(X(t) \in dz) P_z(\xi(0) > s-t, \inf_{u \geq 0} X(u) > y),
\end{aligned}$$

thus

$$P_x(\xi(t) \in ds, I(t) \in dy, X(t) \in dz) = P_x(X(t) \in dz) P_z(\xi(0) \in ds - t, I(0) \in dy).$$

By the lemma,

$$P_z(\xi(0) \in ds - t, I(0) \in dy) = \frac{d-2}{y} P_z(\tau_y \in ds - t)dy,$$

therefore,

$$P_x(\xi(t) \in ds, I(t) \in dy, X(t) \in dz) = \frac{d-2}{y} P_x(X(t) \in dz) P_z(\tau_y \in ds - t)dy.$$

By Kent(1978),

$$P_x(X(t) \in dy)/dy = t^{-1} x^{-\nu} I_\nu(\frac{xy}{t}) \exp(-\frac{1}{2t}(x^2 + y^2)) y^{\nu+1}.$$

When d is an integer,Yin & Wu(1996) proved

$$P_z(\tau_y \in ds - t)/ds = -\frac{1}{\pi}(\frac{y}{z})^\nu \int_0^\infty \frac{\lambda e^{-\frac{1}{2}\lambda^2(s-t)}(J_\nu(\lambda z)N_\nu(\lambda y) - J_\nu(\lambda y)N_\nu(\lambda z))}{J_\nu^2(\lambda y) + N_\nu^2(\lambda y)} d\lambda, \quad (1)$$

where J_ν and N_ν are the first kind and second kind Bessel functions of order ν. Since the P_x distribution of τ_y for any real ν has Laplace transformation (See Kent(1978) or Getoor and Sharpe(1979))

$$E_x e^{-\beta\tau_y} = (\frac{y}{x})^\nu \frac{K_\nu(\sqrt{2\beta}x)}{K_\nu(\sqrt{2\beta}y)}, \quad x \geq y$$

where K_ν is the modified Bessel function. It follows that (1) holds for any positive real number ν.

This completes the proof of the theorem.

Corollary 1 *(1) For any $0 < y \le x$, we have*

$$P_x(I(t) \in \mathrm{d}y)/\mathrm{d}y = \frac{(d-2)y^{d-3}}{tx^\nu} e^{-\frac{x^2}{2t}} \int_y^\infty z^{-\nu+1} I_\nu(\frac{xz}{t}) e^{-\frac{z^2}{2t}} \mathrm{d}z.$$

(2) For $r > 0$, let $\sigma_r = sup\{t > 0, X(t) \le r\}$, then for $x \ge 0$,

$$P_x(\sigma_r \in \mathrm{d}t)/\mathrm{d}t = \frac{\nu r^{2\nu}}{x^\nu t^{\nu+1}} e^{-\frac{x^2}{2}} \int_{\frac{r}{\sqrt{t}}}^\infty y^{-\nu+1} I_\nu(xy) e^{-\frac{y^2}{2}} \mathrm{d}y.$$

Proof (1) Using theorem 1 we have

$$\begin{aligned}
&P_x(I(t) \in \mathrm{d}y)/\mathrm{d}y \\
&= \int_{s=t}^\infty \int_{z=y}^\infty P_x(\xi(t) \in \mathrm{d}s, I(t) \in \mathrm{d}y, X(t) \in \mathrm{d}z) \\
&= \int_{s=t}^\infty \int_{z=t}^\infty \frac{d-2}{y} P_x(X(t) \in \mathrm{d}z) P_z(\tau_y \in \mathrm{d}s - t) \\
&= \int_{z=y}^\infty \frac{d-2}{y} P_x(X(t) \in \mathrm{d}z) \int_{s=t}^\infty P_z(\tau_y \in \mathrm{d}s - t) \\
&= (d-2)y^{d-3} \int_y^\infty (\frac{1}{z})^{d-2} P_x(X(t) \in \mathrm{d}z) \\
&= (d-2)y^{d-3} \int_y^\infty (\frac{1}{z})^{d-2} t^{-1} x^{-\nu} I_\nu(\frac{xz}{t}) \exp(-\frac{x^2+z^2}{2t}) z^{\nu+1} \mathrm{d}z \\
&= \frac{(d-2)y^{d-3}}{tx^\nu} e^{-\frac{x^2}{2t}} \int_y^\infty z^{-\nu+1} I_\nu(\frac{xz}{t}) e^{-\frac{z^2}{2t}} \mathrm{d}z.
\end{aligned}$$

(2) Observe that $(\sigma_r \le t) = (I(t) \ge r)$,thus

$$\begin{aligned}
P_x(\sigma_r \le t) &= P_x(I(t) \ge r) = P_x(I(t) \ge \frac{r}{\sqrt{t}}) = \int_{\frac{r}{\sqrt{t}}}^\infty P_x(I(1) \in \mathrm{d}y) \\
&= \int_{\frac{r}{\sqrt{t}}}^\infty (\frac{(d-2)y^{d-3}}{x^\nu} e^{-\frac{x^2}{2}} \int_y^\infty z^{-\nu+1} I_\nu(xz) e^{-\frac{z^2}{2}} \mathrm{d}z) \mathrm{d}y.
\end{aligned}$$

The result may be obtained by differentiating on t.

References:

[1] GETOOR R K, SHARPE M J. *Excursions of Brownian motion and Bessel processes* [J]. Z. Wahrsch. Verw. Gebiete, 1979, **47**: 83–106.

[2] HU Y, SHI Z. *Sojurns and future infima of planar Brownian motin* [J]. Probab. Th. Rel. Fields, 1995, **103**: 329–348.

[3] KENT J. *Some probabilistic properties of Bessel functions* [J]. Ann. Probab. 1978, **6**: 760–770.

[4] KHOSHMEVISAN D, LEWIS T M. & LI W V. *On the future infima of some transient processes* [J]. Probab. Th. Rel. Fields, 1994, **99**: 337–360.

[5] KHOSHNEVISAN D. *The gap between the past supremum and future infimum of a transient Bessel process,Seminaire de Probabilites XXIX* [J]. Lect. notes in Math., Springer-Berlin, 1995, **1613**: 220–230.

[6] WILLIAMS D. *Decomposing the Brownian path* [J]. Bull. Amer. Math. Soc., 1970, **76**: 871–873.

[7] WILLIAMS D. *Path decmposition and continuity of local time for one-dimensional diffusions* [J]. Proc. London Math. Soc., 1974, **28**: 738–768.

[8] YIN Chuan-cun, WU Rong. *Some problems on balls and spheres for Brownian motion* [J]. Science in China (ser. A), 1996, **39**: 572–582.

[9] ZHAN Shi. *How long does it take a transient Bessel process to reach its future infimum ?* [J]. Lecture Notes in Mathematics, Springer-Berlin, 1996, **1626**: 207–217.

暂留 Bessel 过程的将来极小值及其位置的联合分布

邵 先 喜[1]， 尹 传 存[2]

(1. 青岛大学外贸系，山东 青岛 266071; 2. 曲阜师范大学数学系，山东 曲阜 273165)

摘 要: 对暂留 Bessel 过程 X, 令 $I(t)=\inf_{S\geq t}X(s)$ 及 $\xi(t)=\inf\{u\geq t: X(u)=I(t)\}$, 本文求出了 $I(t),\xi(t)$ 及 X_t 的联合分布.