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On Linearly Topological Structure and Orthogonality of Menger Probabilistic Inner Product Spaces *

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Abstract: In this paper the linearly topological structure of Menger Probabilistic inner product space is discussed. In virtue of these, some more general convergence theorems, Pythagorean theorem, and the orthogonal projective theorem are established.

Key words: propbabilistic inner product space; linear topology; convergence; orthogonal projection.

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1. Introduction

The conceptions of probabilistic inner product space (briefly, PIS) were introduced in [1] , [2], [5]. And some different notions PIS were introduced in [3],[4],[6] successively. We feel that the definition of the PIS in [4] is reasonable, for it keeps in contact with the concept of Menger PN space and the notion of usual real inner product space can be included in much more general setting of such a PIS. It is known that, since t-norm Δ ($\Delta(a,1)=a$) that satisfies $\Delta(t,t)\geq t$ implies $\Delta=\min$, some results will be not general under the condition $\Delta(t,t)\geq t$. The purpose of this paper is to discuss the linearly topological structure of Menger PIS under weaker t-norm conditions, and to establish orthogonal projective theorem in Menger PIS.

2. Preliminaries

Throughout this paper, $\mathcal{D}$ denotes the set of all left-continuous distribution functions; $R=(-\infty,+\infty)$; $R^+=(0,+\infty)$; $\mathcal{D}_0=\{F: F\in\mathcal{D}, F^{-1}(1)\neq\emptyset\}$; $H(t)\in\mathcal{D}_0$: $H(t)=0$ if $t\leq 0$ and $H(t)=1$ if $t>0$; Z^+ stands for the set of all positive integers; "iff" means "if and only if", "a.e." means "almost everywhere".

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Definition 2.1[2] *For a mapping $\Delta : [0,1] \times [0,1] \longrightarrow [0,1]$, Δ is called a t-norm if $\forall a, b, c, d \in [0,1]$:*

(Δ-1) $\Delta(a,1) = a$;

(Δ-2) $\Delta(a,b) = \Delta(b,a)$;

(Δ-3) $c \geq a, d \geq b \Longrightarrow \Delta(c,d) \geq \Delta(a,b)$;

(Δ-4) $\Delta(a, \Delta(b,c)) = \Delta(\Delta(a,b), c)$.

Definition 2.2 (cf. [4]) *A PIS is a ternary $(E, \mathcal{F}, \Delta)$, where E is a real linear space, Δ is a t-norm; $\mathcal{F}$, $E \times E \longrightarrow \mathcal{D}$ is a mapping written by $F_{x,y}$ with $x \in E$ and $y \in E$, $\forall x, y, z \in E$, $(E, \mathcal{F}, \Delta)$ satisfies:*

(PI-1) $F_{x,x}(0) = 0$;

(PI-2) $\forall t \in R$, $F_{x,x}(t) = H(t)$ iff $x = \theta$;

(PI-3) $F_{x,y} = F_{y,x}$;

(PI-4) $\forall h \in R$, $F_{hx,y}(t) = \begin{cases} F_{x,y}\left(\frac{t}{h}\right) & (h > 0), \\ H(t) & (h = 0), \\ 1 - F_{x,y}\left(\frac{t}{h}+\right) & (h < 0); \end{cases}$

(PI-5) $F_{x+y,z}(t) = \sup\limits_{s+r=t; s,r \in R} \Delta\left(F_{x,z}(s), F_{y,z}(r)\right)$.

Example 2.1 (cf.[4]) Let $(E, (\cdot,\cdot))$ be a usual real inner product space, $\mathcal{F}$ be defined by $F_{x,y}(t) = H(t - (x,y))$, Δ be a t-norm. Then $(E, \mathcal{F}, \Delta)$ is a PIS.

Definition 2.3 (cf.[4]) *Let $(E, \mathcal{F}, \Delta)$ be a PIS, $x, y \in E$. If $F_{x,y}(t) = H(t)$ $(\forall t \in R)$, then x and y are said to be arthogonal and written $x \perp y$. Similarly for $M \subset E$ we write $x \perp M$ if $x \perp y$ for $\forall y \in M$.*

Definition 2.4 *Let $(E, \mathcal{F}, \Delta)$ be a PIS. If*

(PI-6) $\forall x, y \in E$ imply $F_{x,y}(ts) \geq \Delta\left(F_{x,x}(t^2), F_{y,y}(s^2)\right)$ a.e. in R^+, then $(E, \mathcal{F}, \Delta)$ is called a Menger PIS . If

(PI-6s) $\forall x, y \in E$, $\forall t, s \in R^+$, $F_{x,y}(ts) \geq \Delta\left(F_{x,x}(t^2), F_{y,y}(s^2)\right)$, then $(E, \mathcal{F}, \Delta)$ is called a strong Menger PIS .

Example 2.2 Let $(E, \mathcal{F}, \Delta)$ be a PIS , then (i) $\Delta = \min \Longrightarrow (E, \mathcal{F}, \Delta)$ is a Menger PIS . (ii) $\Delta =$ product $\Longrightarrow (E, \mathcal{F}, \Delta)$ is a Menger PIS.

Proof Set $h = -t/s$ $(h < 0)$, $a = F_{x,x}(t^2)$, $d = F_{x,y}(ts+)$, $b = F_{x,hy}(hts) = 1 - d$, $c = F_{hy,hy}(h^2 s^2) = F_{y,y}(s^2)$. By (PI-5) we have $0 = F_{x+hy,x+hy}((t+hs)^2) \geq \Delta(\Delta(a,b), \Delta(b,c))$. Then,

(i) $\Delta = \min \Longrightarrow \min(a, 1-d, c) = 0 \Longrightarrow d \geq \min(a,c)$;

(ii) $\Delta =$ product $\Longrightarrow a(1-d)^2 c = 0 \Longrightarrow d \geq ac$.

Since a distribution function is almost everywhere continuous , the proof is complete.

Example 2.3 (cf.[4]) Let $(E, \mathcal{F}, \Delta)$ be a strong Menger PIS, $\Delta = \min$, f: $E \longrightarrow \mathcal{D}$ be a mapping defined by $f_x(t) = 0$ if $t \leq 0$ and $f_x(t) = F_{x,x}(t^2)$ if $t > 0$. Then (E, f, Δ) is Menger PN space.

3. Main results

Lemma 3.1 *Let $(E,\mathcal{F},\Delta)$ be a PIS. $\alpha\in(0,1]$, $\varepsilon>0$, $N(\varepsilon,\alpha)=\{x\in E: F_{x,x}(\varepsilon^2)>1-\alpha\}$. Then*

(i) $N(\varepsilon,\alpha)=\varepsilon N(1,\alpha)$; (ii) $\varepsilon_1\le\varepsilon_2\Longrightarrow N(\varepsilon_1,\alpha)\subset N(\varepsilon_2,\alpha)$; (iii) $\alpha_1\le\alpha_2\Longrightarrow N(\varepsilon,\alpha_1)\subset N(\varepsilon,\alpha_2)$.

Proof These follow from Definition 2.2 .

Theorem 3.2 *Let $(E,\mathcal{F},\Delta)$ be a Menger PIS, $\sup\limits_{b<1}\Delta(b,b)=1$. Then $(E,\mathcal{F},\Delta)$ is a first-countable Hausdorff linear topological space whose neighbourhood base of origin is $\{N(\varepsilon,\alpha):\varepsilon>0,\alpha\in(0,1]\}$.*

Proof (cf. [7]) We will prove the following (i-vi) .

(i) By Lemma 3.1, $\forall W_1=N(\varepsilon_1,\alpha_1)$ and $W_2=N(\varepsilon_2,\alpha_2)$, where $\alpha_0=\{\varepsilon_1,\alpha_1,\varepsilon_2,\alpha_2\}$: $W_0\subset W_1\bigcap W_2$.

(ii) For $\forall W=N(\varepsilon,\alpha)$, by $\sup\limits_{b<1}\Delta(b,b)=1$, $\exists\alpha_1\in(0,\alpha],\alpha_2\in(0,\alpha_1],\alpha_3\in(0,\alpha_2]$: $\Delta(1-\alpha_1,1-\alpha_1)>1-\alpha$, $\Delta(1-\alpha_2,1-\alpha_2)>1-\alpha_1$, and $\Delta(1-\alpha_3,1-\alpha_3)>1-\alpha_2$. By (PI-6), $\exists\delta\in[\varepsilon/4,\varepsilon/2]$: $F_{x,y}(\delta^2)\ge\Delta(F_{x,x}(\delta^2),F_{y,y}(\delta^2))$. Hence, $\exists W_1=N(\varepsilon/4,\alpha_3)$, $\forall x,y\in W_1$: $F_{x,x}(\varepsilon^2/4)\ge F_{x,x}(\varepsilon^2/16)>1-\alpha_3\ge1-\alpha_2$,
Likewise, $F_{y,y}(\varepsilon^2/4)\ge1-\alpha_2$, and

$$\begin{aligned}F_{x,y}(\varepsilon^2/4)&\ge F_{x,y}(\delta^2)\ge\Delta(F_{x,x}(\delta^2),(F_{y,y}(\delta^2))\ge\Delta(F_{x,x}(\varepsilon^2/16),F_{y,y}(\varepsilon^2/16))\\&\ge\Delta(1-\alpha_3,1-\alpha_3)\ge1-\alpha_2.\end{aligned}$$

Thus, by (PI-5) and (Δ-3) ,

$$\begin{aligned}F_{x+y,x+y}(\varepsilon^2)&\ge\Delta\left(\Delta(F_{x,x}(\varepsilon^2/4),F_{x,y}(\varepsilon^2/4)),\Delta(F_{x,y}(\varepsilon^2/4),F_{y,y}(\varepsilon^2/4))\right)\\&\ge\Delta(\Delta(1-\alpha_2,1-\alpha_2),\Delta(1-\alpha_2,1-\alpha_2))\ge\Delta(1-\alpha_1,1-\alpha_1)>1-\alpha,\end{aligned}$$

showing that $W_1+W_1\subset W$.

(iii) For $\forall W=N(\varepsilon,\alpha)$, $\forall\lambda\in R$, $|\lambda|\le1$: if $\lambda=0$, then $0W=\{\theta\}\subset W$; if $\lambda\ne0$, then $\lambda W=N(|\lambda|\varepsilon,\alpha)\subset W$ (Lemma 3.1).

(iv) For $\forall W=N(\varepsilon,\alpha),\forall x\in E$: since $\lim\limits_{t\to\infty}F_{x,x}(t)=1$, $\exists t_0,F_{x,x}(t_0^2)>1-\alpha$. Set $\lambda=\varepsilon/t_0$, then $F_{\lambda x,\lambda x}(\varepsilon^2)=F_{x,x}(t_0^2)>1-\alpha$, namely $\lambda x\in W$.

(v) By (PI-2), if $\forall x\in E$, $x\ne\theta$, then $\exists\lambda_0\in(0,1]$, $\exists\varepsilon_0>0$: $F_{x,x}(\varepsilon_0^2)\le1-\alpha_0$, showing that $x\notin N(\varepsilon_0,\alpha_0)$.

(vi) $\{N(\varepsilon,\alpha):\varepsilon>0,\alpha\in(0,1],\varepsilon$ and α are rational numbers$\}$ is countable .

Corollary 3.3 *If $\Delta=min$, then strong Menger PIS $(E,\mathcal{F},\Delta)$ is a locally convex Hausdorff linear topological space .*

Proof By Theorem 3.2 and Lemma 3.1, we will only prove $N(1,\alpha)$ is convex. In fact, $\forall x,y\in N(1,\alpha)$, $\forall\lambda\in[0,1]$: set $a=F_{\lambda x,\lambda x}(\lambda^2)=F_{x,x}(1)$, $b=F_{\lambda x,(1-\lambda)y}(\lambda(1-\lambda))=F_{x,y}(1)$, $c=F_{(1-\lambda)y,(1-\lambda)y}((1-\lambda)^2)=F_{y,y}(1)$. Since $\Delta=\min$, by (PI-6s) we have $b\ge\min(a,c)$. Hence $F_{\lambda x+(1-\lambda)y,\lambda x+(1-\lambda)y}(1^2)=F_{\lambda x+(1-\lambda)y,\lambda x+(1-\lambda)y}((\lambda+(1-\lambda))^2)\ge\min(\min(a,b),\min(b,c))=\min\{a,c,b\}\ge min\{a,c\}>1-\alpha$. Namely $\lambda x+(1-\lambda)y\in$

$N(1,\alpha)$.

Definition 3.1 *Let $(E,\mathcal{F},\Delta)$ be a Menger PIS, $\sup_{b<1}\Delta(b,b)=1$.*

(i) A sequence $\{x_n\}\subset E$ is said to converge to $x\in E$, denoted by $\lim_{n\to\infty}x_n=x$, if $\forall\varepsilon>0,\forall\alpha\in(0,1]$, $\exists N\in Z^+,\forall n\geq N$, $F_{x_n-x,x_n-x}(\varepsilon^2)>1-\alpha$ (equivalently $\lim_{n\to\infty}F_{x_n-x,x_n-x}(t)=H(t)$).

(ii) A sequence $\{x_n\}\subset E$ is called a Cauchy sequence if $\forall\varepsilon>0$, $\forall\alpha\in(0,1]$, $\exists N\in Z^+$, $\forall m,n\geq N$, $F_{x_m-x_n,x_m-x_n}(\varepsilon^2)>1-\alpha$ (equivalently $\lim_{m,n\to\infty}F_{x_m-x_n,x_m-x_n}(t)=H(t)$); $A\subset E$ is said to be complete if every Cauchy sequence in A converges in A.

Theorem 3.4 *Let $(E,\mathcal{F},\Delta)$ be a Menger PIS, $\forall a\in(0,1]$, $\sup_{b<1}\Delta(b,a)=a$ (implying $\sup_{b<1}\Delta(b,b)=1$), $\lim_{n\to\infty}x_n=x$. Then*

(i) $\forall y\in E$, $\lim_{n\to\infty}F_{x_n,y}(t)=F_{x,y}(t)$ a.e.; *(ii)* $\lim_{n\to\infty}F_{x_n,x_n}(t)=F_{x,x}(t)$ a.e. .

Proof Since $\forall y\in E$, $\lim_{t\to\infty}F_{y,y}(t)=1$, we have

$$\forall\alpha\in(0,1],\exists t_0>0: F_{y,y}(t_0^2)>1-\alpha. \tag{1}$$

By (PI-6), $\forall\varepsilon>0,\exists\delta\in(0,\varepsilon)$, and we have $F_{x_n-x,y}(\delta)\geq\Delta(F_{x_n-x,x_n-x}(\delta^2/t_0^2),F_{y,y}(t_0^2))$. Letting $n\to\infty$, by $\lim_{n\to\infty}F_{x_n-x,x_n-x}(\delta^2/t_0^2)=1$ and $\sup_{b<1}\Delta(b,a)=a$, we have

$$\liminf_{n\to\infty}F_{x_n-x,y}(\delta)\geq F_{y,y}(t_0^2). \tag{2}$$

Since $F_{x_n-x,y}(t)$ is nondecreasing function on R,(1) and (2) imply that

$$\lim_{n\to\infty}F_{x_n-x,y}(\varepsilon)=1. \tag{3}$$

Likewise

$$\lim_{n\to\infty}F_{x-x_n,y}(\varepsilon)=1. \tag{4}$$

Note that $F_{x,y}(t)$ is left-continuous. By (PI-5), (3) and $\sup_{b<1}\Delta(b,a)=a$, we obtain $F_{x_n,y}(t)=F_{x_n-x+x,y}(t)\geq\Delta(F_{x_n-x,y}(\varepsilon),F_{x,y}(t-\varepsilon))$

$$\liminf_{n\to\infty}F_{x_n,y}(t)\geq F_{x,y}(t). \tag{5}$$

Now without loss of generality we may assult $P=\limsup_{n\to\infty}F_{x_n,y}(t)>0$, then $\forall\eta\in(0,P)$, $\exists$ subsequence $\{x_{n_k}\}\subset\{x_n\}$, $\forall k\in Z^+$: $F_{x_{n_k},y}(t)\geq P-\eta$. By (PI-5), (4) and $\sup_{b<1}\Delta(b,a)=a$, we get

$$F_{x,y}(t+\varepsilon)=F_{x-x_{n_k}+x_{n_k},y}(\varepsilon+t)\geq\Delta\left(F_{x-x_{n_k},y}(\varepsilon),F_{x_{n_k},y}(t)\right)\geq\Delta\left(F_{x-x_{n_k},y}(\varepsilon),P-\eta\right),$$

$$\limsup_{n\to\infty}F_{x_n,y}(t)\leq F_{x,y}(t+). \tag{6}$$

Then (5) and (6) show $\lim_{n\to\infty} F_{x_n,y}(t) = F_{x,y}(t)$ a.e.

(ii) For $\forall\varepsilon > 0$, since

$$F_{x_n-x,x_n}(\varepsilon) = F_{x_n-x,x_n-x+x}(\varepsilon) \geq \Delta\left(F_{x_n-x,x_n-x}(\varepsilon/2), F_{x_n-x,x}(\varepsilon/2)\right),$$

by (3) and $\sup_{b<1} \Delta(b,b) = 1$, we have

$$\lim_{n\to\infty} F_{x_n-x,x_n}(\varepsilon) = 1. \tag{7}$$

Likewise

$$\lim_{n\to\infty} F_{x-x_n,x_n}(\varepsilon) = 1. \tag{8}$$

Using Theorem 3.4(i), we have $\exists\delta \in (0,\varepsilon)$:

$$\lim_{n\to\infty} F_{x,x_n}(t-\delta) = F_{x,x}(t-\delta); \tag{9}$$

$$\lim_{n\to\infty} F_{x,x_n}(t+\delta) = F_{x,x}(t+\delta). \tag{10}$$

There is no harm in supposing $L = F_{x,x}(t-\delta) > 0$. Then by (9), $\forall\eta \in (0,L), \exists N \in Z^+$, $\forall n \geq N$: $F_{x,x_n}(t-\delta) > L-\eta$. By (PI-5), (7) and $\sup_{b<1} \Delta(b,a) = a$, we obtain

$$F_{x_n,x_n}(t) = F_{x_n-x+x,x_n}(\delta+t-\delta) \geq \Delta\left(F_{x_n-x,x_n}(\delta), F_{x,x_n}(t-\delta)\right) \geq \Delta\left(F_{x_n-x,x_n}(\delta), L-\eta\right),$$

$$\liminf_{n\to\infty} F_{x_n,x_n}(t) \geq F_{x,x}(t-\delta). \tag{11}$$

There is no harm in supposing $Q = \limsup_{n\to\infty} F_{x_n,x_n}(t) > 0$. Then $\forall\eta \in (0,Q)$, $\exists$ subsequence $\{x_{n_k}\} \subset x_n$, $\forall k \in Z^+$: $F_{x_{n_k},x_{n_k}}(t) > Q-\eta$. By (PI-5), (8), (10) and $\sup_{b<1} \Delta(b,a) = a$ we get

$$\begin{aligned} F_{x,x_{n_k}}(t+\delta) = F_{x-x_{n_k}+x_{n_k},x_{n_k}}(\delta+t) &\geq \Delta(F_{x-x_{n_k},x_{n_k}}(\delta), F_{x_{n_k},x_{n_k}}(t)) \\ &\geq \Delta(F_{x-x_{n_k},x_{n_k}}(\delta), Q-\eta), \end{aligned}$$

$$\limsup_{n\to\infty} F_{x_n,x_n}(t) \leq F_{x,x}(t+\eta). \tag{12}$$

Letting $\varepsilon \to 0$, then $\delta \to 0$, (11) and (12) show $\lim_{n\to\infty} F_{x_n,x_n}(t) = F_{x,x}(t)$ a.e. .

Theorem 3.5 *Let $(E,\mathcal{F},\Delta)$ be a PIS and $\mathcal{F}$ map E into $\mathcal{D}_0$, an inner product on E be defined by $\langle x,y\rangle = \inf\{t : F_{x,y}(t) = 1\}$. Then $(E,\langle\cdot,\cdot\rangle)$ is a classical inner product space, and $x \perp y$ iff $\langle x,y\rangle = 0$.*

Proof (cf. [8]) (IP1) We set $\alpha = \langle x,z\rangle$, $\beta = \langle y,z\rangle$, $\eta = \langle x+y,z\rangle$, $\forall\varepsilon > 0$. Since $F_{x,z}(\alpha + \frac{\varepsilon}{2}) = 1$ and $F_{y,z}(\beta+\frac{\varepsilon}{2}) = 1$, by (PI-5), $F_{x+y,z}(\alpha+\beta+\varepsilon) \geq \Delta\left(F_{x,z}(\alpha+\frac{\varepsilon}{2}), F_{y,z}(\beta+\frac{\varepsilon}{2})\right)$. Namely $F_{x+y,z}(\alpha+\beta+\varepsilon) = 1$. Hence,

$$\eta \leq \alpha+\beta+\varepsilon. \tag{13}$$

On the other hand, since (Δ-1, 2, 3)$\Longrightarrow \Delta \le \min$,

$$1 = F_{x+y,z}(\eta+\varepsilon) = \sup_{s+t=\eta+\varepsilon; s,t\in R} \Delta(F_{x,z}(s), F_{y,z}(t)) \le \sup_{t\in R} \min(F_{x,z}(\eta+\varepsilon-t), F_{y,z}(t)).$$

Then,

$$\forall n \in Z^+, \exists t_n \in R : F_{x,z}(\eta+\varepsilon-t_n) > 1-\frac{1}{n}, \quad F_{y,z}(t_n) > 1-\frac{1}{n}. \tag{14}$$

Hence, $\{t_n\}$ is a bounded real sequence. Otherwise if $\{t_n\}$ is unbounded, we will have $F_{x,z}(-\infty)=1$ or $F_{y,z}(-\infty)=1$ by (14), this provides a contradiction. Thus $\exists$ subsequence $\{t_{n_k}\} \subset \{t_n\}$: $\lim\limits_{k\to\infty} t_{n_k} = t_0$, $|t_0| < +\infty$. By (14), $\exists N \in Z^+$, $\forall k \ge N$: $F_{y,z}(t_0+\varepsilon) \ge F_{y,z}(t_{n_k}) > 1-\frac{1}{n_k}$; $F_{x,z}(\eta+2\varepsilon-t_0) \ge F_{x,z}(\eta+\varepsilon-t_{n_k}) > 1-\frac{1}{n_k}$. Letting $k\to\infty$, we get that $F_{y,z}(t_0+\varepsilon)=1$ and $F_{x,z}(\eta+2\varepsilon-t_0)=1$. Namely

$$t_0+\varepsilon \ge \beta, \eta+2\varepsilon-t_0 \ge \alpha \Longrightarrow \alpha+\beta \le \eta+3\varepsilon \tag{15}$$

By (13) and (15) we obtain $\alpha+\beta=\eta$.

(IP2) If $h=0$, by (PI-4), $F_{\theta,y}(t)=H(t)$, i.e., $\langle\theta,y\rangle=0$, then $\langle\theta x,y\rangle=\langle\theta,y\rangle=0=0\langle x,y\rangle$; if $h>0$, by (PI-4), $F_{hx,y}(t)=F_{x,y}\left(\frac{t}{h}\right)$, then $\langle hx,y\rangle=\inf\{t : F_{x,y}\left(\frac{t}{h}\right)=1\}=h\langle x,y\rangle$; if $h<0$, by (IP1), $\langle hx,y\rangle+\langle -hx,y\rangle=\langle\theta,y\rangle=0$, then $\langle hx,y\rangle=-\langle -hx,y\rangle=h\langle x,y\rangle$.

The proof of (IP3) and (IP4) are routine. Finally, $x\perp y$ iff $F_{x,y}(t)=H(t)$ iff $\langle x,y\rangle=0$.

Theorem 3.6 (Pythagorean theorem) *Let $(E,\mathcal{F},\Delta)$ be a PIS, $x\perp y$. Then*

$$F_{x+y,x+y}(t) = \sup_{s+r=t; s,r\in R} \Delta(F_{x,x}(s), F_{y,y}(r)).$$

Proof By $F_{x,y}(t)=H(t)$, (PI-5) and (Δ-1) we have

$$\begin{aligned} F_{x,x+y}(s) &= \sup_{u+v=s; u,v\in R} \Delta(F_{x,x}(u), F_{x,y}(v)) = \sup_{v>0} \Delta(F_{x,x}(s-v), H(v)) \\ &= \sup_{v>0} F_{x,x}(s-v) = F_{x,x}(s). \end{aligned}$$

Similarly $F_{y,x+y}(r)=F_{y,y}(r)$. Hence,

$$F_{x+y,x+y}(t) = \sup_{s+r=t; s,r\in R} \Delta(F_{x,x+y}(s), F_{y,x+y}(r)) = \sup_{s+r=t; s,r\in R} \Delta(F_{x,x}(s), F_{y,y}(r)).$$

Theorem 3.7 (projective theorem) *Let M be a complete linear subspace of a Menger PIS $(E,\mathcal{F},\Delta)$, $\forall a\in[0,1]$, $\sup\limits_{b<1}\Delta(b,a)=a$, $\mathcal{F}$ map E into $\mathcal{D}_0$, $x\in E$. Then $\exists$ unique $x_0\in M$: $(x-x_0)\perp M$ (x_0 is called the orthogonal projection of x on M) and*

$$F_{x-x_0,x-x_0}(t) = \sup_{y\in M} F_{x-y,x-y}(t) \tag{16}$$

Proof If $\exists x_0 \in M$, $(x-x_0) \perp M$, then $\forall y \in M$, $x-y=(x-x_0)+(x_0-y)$. Since $(x_0-y) \in M$, $(x-x_0) \perp (x_0-y)$, using Theorem 3.6,

$$\begin{aligned} F_{x-y,x-y}(t) &= \sup_{s+r=t;s,r\in R} \Delta(F_{x-x_0,x-x_0}(s), F_{x_0-y,x_0-y}(r)) \\ &\le \sup_{s+r=t,s,r\in R} \Delta(F_{x-x_0,x-x_0}(s), H(r)) \\ &= \sup_{r>0} F_{x-x_0,x-x_0}(t-r) = F_{x-x_0,x-x_0}(t). \end{aligned}$$

Note that $x_0 \in M$, $F_{x-x_0,x-x_0}(t) \in \{F_{x-y,x-y}(t) : y \in M\}$. Thus (16) holds. Applying Theorem 3.5 and the projective theorem of the classical inner product space, we will only prove that M is complete in $(E, \langle\cdot,\cdot\rangle)$. Suppose $\{x_n\} \subset M$ is a Cauchy sequence in $(E, \langle\cdot,\cdot\rangle)$. Then $\forall \varepsilon > 0, \exists N \in Z^+, \forall m,n \ge N$, $\langle x_n-x_m, x_n-x_m\rangle = \inf\{t : F_{x_n-x_m,x_n-x_m}(t) = 1 < \varepsilon^2\}$, namely

$$F_{x_n-x_m,x_n-x_m}(\varepsilon^2) = 1 \tag{17}$$

Thus $\{x_n\} \subset M$ is also a Cauchy sequence in $(E, \mathcal{F}, \Delta)$. Since M is complete in $(E, \mathcal{F}, \Delta)$, it shows that $\lim_{n\to\infty} x_n = x \in M$. Using Theorem 3.4 , $\exists t_0 \in [\varepsilon^2, 2\varepsilon^2]$, from (17), we have $F_{x_n-x,x_n-x}(t_0) = \lim_{m\to\infty} F_{x_n-x_m,x_n-x_m}(t_0) = 1$. Namely $\forall \varepsilon > 0, \exists N \in Z^+$, $\forall n \ge N$, $\langle x_n-x, x_n-x\rangle < t_0 \le 2\varepsilon^2$. This completes the proof .

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关于 Menger 概率内积空间的线性拓扑结构和正交性质

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摘 要: 本文讨论了 Menger 概率内积空间的线性拓扑结构，建立了空间上较为一般的收敛定理，勾股定理及正交投影定理等.