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Several Derivatives of Adjoined Distributions *

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Abstract: A representation theorem for $(x \pm i0)^{\lambda} \ln^{k}(x \pm i0)$ is proved and then the derivatives $(\ln^{k} x_{\pm})', (x_{\pm}^{\lambda} \ln^{k} x_{\pm})', (x_{\pm}^{-n} \ln^{k} x_{\pm})', (\mathrm{d}/\mathrm{d}x)\{(x \pm i0)^{\lambda} \ln^{k}(x \pm i0)\}$ and $(\mathrm{d}/\mathrm{d}x)\{(x \pm i0)^{-n} \ln^{k}(x \pm i0)\}$ are given.

Key words: representation theorem; the regularization of the divergent integral.

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Let's denote by I the set of all integers and let

$$I_{+} = \{i \in I | i > 0\}, I_{-} = \{i \in I | i < 0\}, I_{\pm}^{0} = \{0\} \cup I_{\pm}.$$

Let C denotes the field of complex numbers.

The following adjoined distributions were defined in [1], [2]

$$\ln(x \pm i0) = \lim_{y \to 0_{+}} \ln(x \pm iy) = \ln|x| \pm i\pi H(-x), \tag{1}$$

$$(x \pm i0)^{\lambda} \ln^{k}(x \pm i0) = (\partial^{k}/\partial\lambda_{1}^{k})(x \pm i0)^{\lambda_{1}}|_{\lambda_{1}=\lambda}, \tag{2}$$

$$(x \pm i0)^{-n} \ln^{k}(x \pm i0) = \lim_{\lambda \to -n} (\partial^{k}/\partial\lambda^{k})(x_{+}^{\lambda} + e^{\pm i\lambda\pi} x_{-}^{\lambda}) \tag{3}$$

for $\lambda \in C \backslash I_{-}$ and $k, n \in I_{+}, H(x)$ denotes Heaviside's function. Further we have[2]

$$(x \pm i0)^{0} \ln^{k}(x \pm i0) = \ln^{k} x_{+} + \sum_{j=0}^{k} \binom{k}{j} (\pm i\pi)^{k-j} \ln^{j} x_{-} = \ln^{k}(x \pm i0) \tag{4}$$

for $k \in I_{+}$ where $\ln^{0} x_{-} = H(-x)$.

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Theorem 1 (Representation theorem) *The distributions* $(x \pm i0)^{\lambda} \ln^k(x \pm i0)$ *are entire functions in* λ *and*

$$(x \pm i0)^{\lambda} \ln^k(x \pm i0) = x_+^{\lambda} \ln^k x_+ + e^{\pm i\lambda\pi} \sum_{j=0}^{k} \binom{k}{j} (\pm i\pi)^{k-j} x_-^{\lambda} \ln^j x_-, \tag{5}$$

$$(x \pm i0)^{-n} \ln^k(x \pm i0) = x_+^{-n} \ln^k x_+ + (-1)^n \sum_{j=0}^{k} \binom{k}{j} (\pm i\pi)^{k-j} x_-^{-n} \ln^j x_- + (-1)^n (\pm i\pi)^{k+1} \delta^{(n-1)}(x)/\{(k+1)(n-1)!\} \tag{6}$$

for $\lambda \in C\backslash I_-, k \in I_+^0$ *and* $n \in I_+$ *where we assume that*

$$x_\pm^{\lambda} \ln^0 x_\pm = x_\pm^{\lambda}, \quad (x \pm i0)^{\lambda} \ln^0(x \pm i0) = (x \pm i0)^{\lambda},$$

$$x_\pm^{-n} \ln^0 x_\pm = x_\pm^{-n}, \quad (x \pm i0)^{-n} \ln^0(x \pm i0) = (x \pm i0)^{-n}.$$

Proof The distributions $(x \pm i0)^{\lambda}$ are entire functions in the variable λ, and so $(x \pm i0)^{\lambda} \ln^k(x \pm i0)$ with $k \in I_+$ are also entire functions by differentiating the functions $(x \pm i0)^{\lambda}$ in λ with k times respectively.

We write the Taylor-series expansions of the functions $(x \pm i0)^{\lambda}, x_\pm^{\lambda}$ and $e^{\pm i\lambda\pi}$ in powers of $\lambda - \lambda_0$ on a neighborhord of the point $\lambda = \lambda_0$.

$$(x \pm i0)^{\lambda} = \sum_{j=0}^{\infty} (\lambda - \lambda_0)^j (x \pm i0)^{\lambda_0} \ln^j(x \pm i0)/j!,$$

$$x_\pm^{\lambda} = \sum_{j=0}^{\infty} (\lambda - \lambda_0)^j x_\pm^{\lambda_0} \ln^j x_\pm/j!,$$

$$e^{\pm i\lambda\pi} = \sum_{j=0}^{\infty} (\lambda - \lambda_0)^j e^{\pm i\lambda_0\pi} (\pm i\pi)^j/j!$$

for $\lambda \in C\backslash I_-$, by substituting the above expansions into the relation

$$(x \pm i0)^{\lambda} = x_+^{\lambda} + e^{\pm i\lambda\pi} x_-^{\lambda} \tag{7}$$

the equality (5) follows by comparing the coefficients of the terms with the same degree in (7) and then replacing λ_0 by λ.

To prove the equality (6) we write the Taylor-series expansions of the functions $(x \pm i0)^{\lambda}$ and $e^{\pm i\lambda\pi}$, and the Laurent-series expansions of the functions x_+^{λ} and x_-^{λ}, in powers of $(\lambda + n)$ on a neighborhood of the point $\lambda = -n$

$$(x \pm i0)^{\lambda} = \sum_{j=0}^{\infty} (\lambda + n)^j (x \pm i0)^{-n} \ln^j(x \pm i0)/j!,$$

$$e^{\pm i\lambda\pi} = \sum_{j=0}^{\infty} (-1)^n (\lambda + n)^j (\pm i\pi)^j/j!,$$

$$x_\pm^{\lambda} = (\mp 1)^{n-1} (\lambda + n)^{-1} \delta^{(n-1)}(x)/(n-1)! + \sum_{j=0}^{\infty} (\lambda + n)^j x_\pm^{-n} \ln^j x_\pm/j!$$

for $n \in I_+$, by substituting the above expansions into the relation (7) the equality (6) follows by comparing the coefficients of the terms with the same degree in (7). This completes the proof.

Theorem 2 *Let $k \in I_+$. Then*

$$(\mathrm{d}/\mathrm{d}x)(\ln^k x_+) = kx_+^{-1}\ln^{k-1} x_+. \tag{8}$$

Proof If $k=1$, (8) becomes $(\mathrm{d}/\mathrm{d}x)(\ln x_+) = x_+^{-1}$ which was proved in [1]. Suppose that $k>1$, for any test function $\Phi \in \mathcal{D}$ we have

$$\begin{aligned}\langle(\ln^k x_+)', \Phi\rangle &= -\langle \ln^k x_+, \Phi'\rangle = -\int_0^\infty \ln^k x\Phi'(x)\mathrm{d}x \\ &= -\lim_{\varepsilon\to 0}\int_\varepsilon^\infty \ln^k x \mathrm{d}\Phi(x) = \lim_{\varepsilon\to 0}\{\Phi(0)\ln^k\varepsilon + \int_\varepsilon^\infty kx^{-1}\ln^{k-1}x\Phi(x)\mathrm{d}x\}.\end{aligned}$$

by using that $\ln^k\varepsilon\{\Phi(\varepsilon)-\Phi(0)\}\to 0$ as $\varepsilon\to 0$. Further

$$\begin{aligned}\Phi(0)\ln^k\varepsilon &= -\Phi(0)\int_\varepsilon^1 \mathrm{d}(\ln^k x) = -\Phi(0)\int_\varepsilon^1 kx^{-1}\ln^{k-1}x\mathrm{d}x \\ &= -\int_\varepsilon^\infty kx^{-1}\ln^{k-1}x\Phi(0)H(1-x)\mathrm{d}x,\end{aligned}$$

where $H(x)$ denotes Heaviside's function. Consequently

$$\langle(\ln^k x_+)', \Phi\rangle = \int_0^\infty kx^{-1}\ln^{k-1}x[\Phi(x)-\Phi(0)H(1-x)]\mathrm{d}x = \langle kx_+^{-1}\ln^{k-1}x_+, \Phi\rangle$$

by the regularization of the divergent integral (see [1]).

Corollary 1 *Let $k \in I_+$. Then*

$$(\mathrm{d}/\mathrm{d}x)(\ln^k x_-) = -kx_-^{-1}\ln^{k-1}x_-, \tag{9}$$

$$(\mathrm{d}/\mathrm{d}x)(\ln^k |x|) = kx^{-1}\ln^{k-1}|x|. \tag{10}$$

Theorem 3 *Let $\lambda \in C\backslash I_-^0$ and $k \in I_+$. Then*

$$(\mathrm{d}/\mathrm{d}x)(x_+^\lambda \ln^k x_+) = \lambda x_+^{\lambda-1}\ln^k x_+ + kx_+^{\lambda-1}\ln^{k-1}x_+. \tag{11}$$

Proof If $\mathrm{Re}\lambda > 0$ then for any $\Phi(x)\in\mathcal{D}$ we have

$$\langle x_+^\lambda\ln^k x_+, \Phi'(x)\rangle = -\langle \lambda x_+^{\lambda-1}\ln^k x_+ + kx_+^{\lambda-1}\ln^{k-1}x_+, \Phi(x)\rangle$$

for $k\in I_+$. This means that (11) is true for $\mathrm{Re}\lambda>0$. By the analytical continuation the equality (11) holds on the whole complex plane except $\lambda = 0,-1,-2,\cdots$ in virture of the uniqueness of the analytic continuation.

Corollary 2 *The equality*

$$(\mathrm{d}/\mathrm{d}x)(x_-^\lambda\ln^k x_-) = -\lambda x_-^{\lambda-1}\ln^k x_- - kx_-^{\lambda-1}\ln^{k-1}x_- \tag{12}$$

holds for $\lambda \in C\backslash I_-^0$ and $k \in I_+$. The equality

$$(\mathrm{d}/\mathrm{d}x)(|x|^\lambda \ln^k |x|) = \lambda |x|^{\lambda-1} \ln^k |x| \mathrm{sgn} x + k|x|^{\lambda-1} \ln^{k-1} |x| \mathrm{sgn} x \tag{13}$$

holds for $\lambda \in C\backslash\{-1,-3,\cdots\}$ and $k \in I_+$. The equality

$$(\mathrm{d}/\mathrm{d}x)(|x|^\lambda \ln^k |x| \mathrm{sgn} x) = \lambda |x|^{\lambda-1} \ln^k |x| + k|x|^{\lambda-1} \ln^{k-1} |x| \tag{14}$$

holds for $\lambda \in C\backslash\{0,-2,-4,\cdots\}$ and $k \in I_+$. In particular we have

$$(\mathrm{d}/\mathrm{d}x)(x^{-n} \ln^k |x|) = -nx^{-n-1} \ln^k |x| + kx^{-n-1} \ln^{k-1} |x| \tag{15}$$

for $n \in I_+$ and $k \in I_+$.

Theorem 4 *Let $n \in I_+$. Then (see [1])*

$$(\mathrm{d}/\mathrm{d}x)(x_+^{-n}) = -nx_+^{-n-1} + (-1)^n \delta^{(n)}(x)/n!, \tag{16}$$

$$(\mathrm{d}/\mathrm{d}x)(x_-^{-n}) = nx_-^{-n-1} - \delta^{(n)}(x)/n!, \tag{17}$$

$$(\mathrm{d}/\mathrm{d}x)(x^{-n}) = -nx^{-n-1}. \tag{18}$$

Theorem 5 *Let $n \in I_+$ and $k \in I_+$. Then*

$$(\mathrm{d}/\mathrm{d}x)(x_+^{-n} \ln^k x_+) = -nx_+^{-n-1} \ln^k x_+ + kx_+^{-n-1} \ln^{k-1} x_+. \tag{19}$$

Proof By the regularization of the divergent integral [1] we have

$$\begin{aligned}
&\langle (\mathrm{d}/\mathrm{d}x)(x_+^{-n} \ln^k x_+), \Phi(x)\rangle = -\langle x_+^{-n} \ln^k x_+, \Phi'(x)\rangle \\
&= -\int_0^\infty x^{-n} \ln^k x \{\Phi'(x) - \Phi'(0) - x\Phi^{(2)}(0) - \cdots - x^{n-1}\Phi^{(n)}(0)H(1-x)/(n-1)!\}\mathrm{d}x \\
&= -\int_0^1 x^{-n} \ln^k x d\{\Phi(x) - \Phi(0) - \cdots - x^n \Phi^{(n)}(0)/n!\} - \\
&\quad \int_1^\infty x^{-n} \ln^k x d\{\Phi(x) - \Phi(0) - \cdots - x^{n-1}\Phi^{(n-1)}(0)/(n-1)!\} \\
&= -\int_0^\infty \{\Phi(x) - \Phi(0) - \cdots - x^n \Phi^{(n)}(0)H(1-x)/n!\}\mathrm{d}(x^{-n}\ln^k x) + \ln^k x|_{x=1} \Phi^{(n)}(0)/n! \\
&= \int_0^\infty (-nx^{-n-1}\ln^k x + kx^{-n-1}\ln^{k-1} x)\{\Phi(x) - \Phi(0) - \cdots - x^n\Phi^{(n)}(0)H(1-x)/n!\}\mathrm{d}x
\end{aligned}$$

for any $\Phi \in \mathcal{D}$. Hence (19) holds.

Corollary 3 *Let $n, k \in I_+$. Then*

$$(\mathrm{d}/\mathrm{d}x)(x_-^{-n} \ln^k x_-) = nx_-^{-n-1} \ln^k x_- - kx_-^{-n-1} \ln^{k-1} x_-, \tag{20}$$

$$(\mathrm{d}/\mathrm{d}x)(x^{-n} \ln^k |x|) = -nx^{-n-1} \ln^k |x| + kx^{-n-1} \ln^{k-1} |x| \tag{21}$$

which agrees with (15).

Theorem 6 *The equality*

$$(\mathrm{d}/\mathrm{d}x)\{\ln^k(x\pm i0)\} = k(x\pm i0)^{-1}\ln^{k-1}(x\pm i0) \tag{22}$$

holds for $k\in I_+$. In particular if $k=1$ we have

$$(\mathrm{d}/\mathrm{d}x)\{\ln(x\pm i0)\} = (x\pm i0)^{-1}. \tag{23}$$

Proof Using (4),(8),(9) and Theorem 1 for $k\in I_+$ we have

$$\begin{aligned}(\mathrm{d}/\mathrm{d}x)\{\ln^k(x\pm i0)\} &= (\ln^k x_+)' + \sum_{j=0}^{k}\binom{k}{j}(\pm i\pi)^{k-j}(\ln^j x_-)' \\ &= kx_+^{-1}\ln^{k-1}x_+ + \sum_{j=1}^{k}\binom{k}{j}(\pm i\pi)^{k-j}(-jx_-^{-1}\ln^{j-1}x_-) - (\pm i\pi)^k\delta(x) \\ &= k\{x_+^{-1}\ln^{k-1}x_+ - \sum_{j=0}^{k-1}\binom{k-1}{j}x_-^{-1}\ln^j x_- - (\pm i\pi)^k\delta(x)/k\} \\ &= k(x\pm i0)^{-1}\ln^{k-1}(x\pm i0).\end{aligned}$$

Similarly the following two theorems can be proved by using (11),(12),(19),(20) and Theorem 1.

Theorem 7 *Let $\lambda\in C\backslash I_-^0$ and $k\in I_+$. Then*

$$(\mathrm{d}/\mathrm{d}x)\{(x\pm i0)^\lambda\ln^k(x\pm i0)\} = \lambda(x\pm i0)^{\lambda-1}\ln^k(x\pm i0) + k(x\pm i0)^{\lambda-1}\ln^{k-1}(x\pm i0). \tag{24}$$

Theorem 8 *Let $k,n\in I_+$. Then*

$$\begin{aligned}&(\mathrm{d}/\mathrm{d}x)\{(x\pm i0)^{-n}\ln^k(x\pm i0)\} \\ &\quad = -n(x\pm i0)^{-n-1}\ln^k(x\pm i0) + k(x\pm i0)^{-n-1}\ln^{k-1}(x\pm i0).\end{aligned} \tag{25}$$

References:

[1] GELFAND I M, SHILOV G E. *Generalized Functions vol.I* [M]. Academic Press, 1964.

[2] FISHER B. *The generalized function* $(x\pm i0)^\lambda$ [J]. Proc. Camb. Phil. Soc., 1970, **68**: 707–708.

附加广义函数的几个导数

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摘 要: 作者证明了广义函数 $(x\pm i0)^\lambda\ln^k(x\pm i0)$ 的表示定理，给出了附加广义函数的导数： $(\ln^k x_\pm)', (x_\pm^\lambda\ln^k x_\pm)', (x_\pm^{-n}\ln^k x_\pm)', (\mathrm{d}/\mathrm{d}x)\{(x\pm i0)^\lambda\ln^k(x\pm i0)\}$ 和 $(\mathrm{d}/\mathrm{d}x)\{(x\pm i0)^{-n}\ln^k(x\pm i0)\}$.