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Geometric Characterizations of Convergence for Sequences of Continuous Linear Functionals *

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Abstract: We prove the following main result: Let X be a normed linear space, $f_n \in X^*\backslash\{\theta\}, H_n = \{x \in X : f_n(x) = 1\}, n = 0, 1, 2, \cdots$. Then $w^* - \lim\limits_n f_n = f_0$ iff $H_0 \subset \liminf\limits_n H_n$ and $\theta \notin \limsup\limits_n H_n$; when X is a reflexive Banach space, $\lim\limits_n \|f_n - f_0\| = 0$. If and only if $\theta \notin w - \limsup\limits_n H_n \subset H_0$. It simplifies the related results in [1].

Key words: norm-(weak-,weak*-) convergence; Kuratowski-(Mosco-, Wijsman-) convergence.

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1. Introduction

In [1], G. Beer proved that when X is a Banach space, weak* convergence of a sequence $\{f_n\} \subset X^*$ to $f_0 \neq \theta$ is equivalent to the Kuratowski convergence of level sets (i.e. , hyperplanes determined by f_n and α)$\{x \in X : f_n(x) = \alpha\}(n = 1, 2, \cdots)$ to $\{x \in X : f_0(x) = \alpha\}$ for each real α, and that when X is reflexive, norm convergence of $\{f_n\}$ in X* to $f_0 \neq \theta$ is equivalent to the Mosco convergence of level sets. Motivated by his work, in the present paper we shall prove, in order to characterize geometrically weak* convergence (resp. norm convergence) for a sequence of non zero continuous linerar functionals on X one merely needs to use one of the two inclusion relations in the definition of Kuratowski convergence (resp. Mosco convergence) for corresponding sequence of level sets. Precisely, let $f_n \in X^*\backslash\{\theta\}, H_n = \{x \in X : f_n(x) = 1\}(n = 0, 1, 2, \cdots)$. When X is a normed linear space , $w^* - \lim\limits_n f_n = f_0$ iff $H_0 \subset \liminf\limits_n H_n$ and $\theta \notin \limsup\limits_n H_n$ (see Theorem 1); and when X is a reflexive Banach space, $\lim\limits_n \|f_n - f_0\| = 0$ iff $\theta \notin w - \limsup\limits_n H_n \subset H_0$ (see Theorem 2). This is an improvement and simplification of the related results in [1].

2. Preliminaries

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Let X be a real normed linear space and X^* be its dual. The origin of the space is denoted by θ. We begin by recalling several notions of convergence for a sequence of sets in X. Let $\{A_n\}$ be a sequence of nonempty subsets of X. Define

$$\liminf_n A_n = \{x \in X : x = \lim_n x_n, x_n \in A_n (n = 1, 2, \cdots)\},$$
$$\limsup_n A_n = \{x \in X : x = \lim_k x_{n_k}, x_{n_k} \in A_{n_k} (k = 1, 2, \cdots)\},$$
$$w - \limsup_n A_n = \{x \in X : x = w - \lim_k x_{n_k}, x_{n_k} \in A_{n_k} (k = 1, 2, \cdots)\}.$$

If $A \subset X$ is satisfies that $A \subset \liminf_n A_n$ and $\limsup_n A_n \subset A$,i.e. , $\liminf_n A_n = A = \limsup_n A_n$, then we say that $\{A_n\}$ Kuratowski converges to A and write $K - \lim_n A_n = A$.

If $A \subset X$ is such that $A \subset \liminf_n A_n$ and $w - \limsup_n A_n \subset A$,i.e., $\liminf_n A_n = A = w - \limsup_n A_n$, then we say that $\{A_n\}$ Mosco converges to A and write $M - \lim_n A_n = A$.

Evidently , $M - \lim_n A_n = A$ implies $K - \lim_n A_n = A$.

If $\lim_n d(x, A_n) = d(x, A)$ for each $x \in X$, where $d(x, A) = \inf\{\| x - a \| : a \in A\}$, then we say that $\{A_n\}$ Wijsman converges to A and write $W - \lim_n A_n = A$.

There are many references for various types of convergence mentioned above (see, for example, [1-6]).

It is well known that [7,8,9], as a particular kind of closed and convex subsets of X, closed hyperplane in X which do not pass through the origin θ are in one-to-one correspondence with the nonzero continuous linear functional on X. This correspondence is given by $H = \{x \in X : f(x) = 1\}$ and is called the characteristic hyperplane of f. Obviously , the characteristic hyperplane is a particular case of the level sets.

For simplicity , we focus our discussion on characteristic hyperplanes . As in [1] , the results obtained in the present paper remain valid for level sets which are the form of $\{x \in X : f(x) = \alpha\}$ where $\alpha \neq 0$.

3. Main results

In this section, H_n will always denote the characteristic hyperplane of a continuous linear functional f_n on X, i.e., $H_n = \{x \in X : f_n(x) = 1\}$,where n is a non-negative integer.

Lemma *Let X be a real normed linear space and $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonzero continuous linear functionals on X. The following statements are equivalent:*

(i) $\{f_n\}$ is norm-bounded;

(ii) $\inf\{d(\theta, H_n) : n \in \mathrm{N}\} > 0$;

(iii) $\theta \notin \limsup_n H_n$.

Proof By Ascoli's Lemma [8,p.24], we have that d $(\theta, H_n) = \frac{1}{\|f_n\|} (n = 1, 2, \cdots)$. thus the equivalence of (i) and (ii), follows immediately.

Next suppose $\theta \in \limsup_n H_n$. Then there exists a sequence $\{x_{n_k}\}, x_{n_k} \in H_{n_k} (k = 1, 2, \cdots)$, such that $x_{n_k} \to \theta, i.e., \| x_{n_k} \| \to 0 (k \to \infty)$.Since $\mathrm{d}(\theta, H_{n_k}) \leq \| x_{n_k} \| \ (k = 1, 2, \cdots)$.Then $\mathrm{d}(\theta, H_{n_k}) \to 0$ as $k \to \infty$. Therefore (ii) implies (iii).

Finally , suppose $\inf\{d(\theta, H_n) : n \in N\} = 0$. Then there exists a subsequence $\{H_{n_k}\}$ of $\{H_n\}$ such that $d(\theta, H_{n_k}) \to 0$ as $k \to \infty$. Thus $\theta \in \lim_n \sup H_n$. Therefore (iii) implies (ii). And so the proof is complete.

Theorem 1 *Let X be a normed linear space, and let $f_n \in X^* \setminus \{\theta\}$ and $H_n = \{x \in X : f_n(x) = 1\}(n = 0, 1, 2, \cdots)$. Suppose $\theta \notin \lim_n \sup H_n$. Then $w^* - \lim_n f_n = f_0$ if and only if $H_0 \subset \lim_n \inf H_n$.*

Proof First assume that $H_0 \subset \lim_n \inf H_n$. Since $\theta \notin \lim_n \sup H_n$, we may assume from the Lemma that $\|f_n\| \leq M$ for each $n \in N$, where M is a positive number.

Suppose $w^* - \lim_n f_n = f_0$ fails. Then there exist $\varepsilon_1 > 0$ and $x_1 \in X \setminus \{\theta\}$, and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that for each $k \in N$,

$$|f_{n_k}(x_1) - f_0(x_1)| \geq \varepsilon_1. \tag{1}$$

if $f_0(x_1) \neq 0$, let

$$x_0 = \frac{1}{f_0(x_1)} x_1, \quad \varepsilon_0 = \frac{1}{|f_0(x_1)|} \varepsilon_1.$$

Then $x_0 \in H_0$ and

$$|f_{n_k}(x_0) - 1| = |f_{n_k}(x_0) - f_0(x_0)| = \frac{1}{|f_0(x_1)|} |f_{n_k}(x_1) - f_0(x_1)| \geq \varepsilon_0.$$

By Ascoli's Lemma, we have

$$d(x_0, H_{n_k}) = \frac{|f_{n_k}(x_0) - 1|}{\|f_{n_k}\|} \geq \frac{\varepsilon_0}{M}. \tag{2}$$

On the other hand , since $x \in \lim_i \inf H_i$ means $\lim_i d(x, H_i) = 0$, the assumption $x_0 \in H_0 \subset \lim_n \inf H_n \subset \lim_k \inf H_{n_k}$ implies $\lim_n d(x_0, H_{n_k}) = 0$, which contradicts to (2).

If $f_0(x_1) = 0$, then (1) becomes simply $|f_{n_k}(x_1)| \geq \varepsilon_1$ for all $k \in N$. Choose $z_0 \in H_0$,and let $y_1 = x_1 - z_0$. Thus

$$\begin{aligned} |f_{n_k}(y_1) - f_0(y_1)| &= |f_{n_k}(x_1) - f_{n_k}(z_0) + f_0(z_0)| \\ &\geq |f_{n_k}(x_1)| - |f_{n_k}(z_0) - 1| \\ &= |f_{n_k}(x_1)| - \|f_{n_k}\| d(z_0, H_{n_k}) \\ &\geq |f_{n_k}(x_1)| - M d(z_0, H_{n_k}). \end{aligned}$$

From the assumption $H_0 \subset \lim_n \inf H_n$ it follows that $\lim_n d(z_0, H_{n_k}) = 0$. Hence there exists $k_0 \in N$ such that for all $k \geq k_0$,

$$|f_{n_k}(y_1) - f_0(y_1)| \geq |f_{n_k}(x_1)| - M d(z_0, H_{n_k}) \geq \frac{\varepsilon_1}{2}. \tag{1'}$$

In the same way as the above proof in the case of $f_0(x_1) \neq 0$ (merely replace x_1 by y_1 and ε_1 by $\frac{\varepsilon_1}{2}$), we obtain again a contradiction. Therefore we conclude that $w^* - \lim_n f_n = f_0$.

Conversely, assume that $w^* - \lim_n f_n = f_0$. We claim that $H_0 \subset \liminf_n H_n$. Let $x \in H_0$, then $f_0(x) = 1$. For each $n \in \mathbf{N}$ let $f_n(x) = \alpha_n$. Since $\lim_n f_n(x) = f_0(x)$ we have $\alpha_n \to 1$ as $n \to \infty$. Without loss of generality , we may assume that $\alpha_n > 0$ for each $n \in \mathbf{N}$. Let $x_n = \frac{1}{\alpha_n}x$, then $f_n(x_n) = \frac{1}{\alpha_n}f_n(x) = 1$, and so $x_n \in H_n (n = 1, 2, \cdots)$. Moreover, $\|x_n - x\| = \|\frac{1}{\alpha_n}x - x\| = \left|\frac{1}{\alpha_n} - 1\right| \|x\| \to 0$ as $n \to \infty$, so that $x \in \liminf_n H_n$.

This completes the proof . □

As an immediate consequence of Theorem 1 and Lemma, we have

Corollary 1 *Suppose $\theta \notin \limsup_n H_n$. Then $K - \lim_n H_n = H_0$ if and only if $H_0 \subset \liminf_n H_n$.*

Corollary 2 *Let X be a normed linear space, and let $f_n \in X^* \backslash \{\theta\}$ and $H_n = \{x \in X : f_n(x) = 1\} (n = 0, 1, 2, \cdots)$. Suppose $\theta \notin \limsup_n H_n$. Then $W - \lim_n H_n = H_0$ if and only if $H_0 \subset \liminf_n H_n$ and $\lim_n d(\theta, H_n) = \mathrm{d}(\theta, H_0)$.*

Remark Even if X is a Banach space the "if" part of Theorem 1 and Corollary 1, may fail without the assumption $\theta \notin \limsup_n H_n$.

Example Let $X = l_2$. For each $n \in \mathbf{N}$ let $f_n = ne_n$ and $f_0 = e_1$, where e_n is the n-th unit vector. It is easily seen that $\{f_n\}$ is (norm) unbounded (*i.e.*, $\theta \in \limsup_n H_n$ by Lemma). By Uniform Boundedness Principle, $\{f_n\}$ is not weak* convergent.

To see that $H_0 \subset \liminf_n H_n$, let $x = (\xi_i)_{i=1}^{\infty} \in H_0$, so that $f_0(x) = \xi_1 = 1$. Then $x = (1, \xi_2, \xi_3, \cdots)$, where $\xi_2, \xi_3, \cdots$ are real numbers such that $\sum_{i=2}^{\infty} |\xi_i|^2 < \infty$. For each $n \in \mathbf{N}$, we can choose $x_n = (1, \xi_2, \cdots, \xi_{n-1}, \frac{1}{n}, \xi_{n+1}, \cdots) \in H_n$, so that

$$\|x_n - x\|_2 = |\frac{1}{n} - \xi_n| \to 0 \text{ as } n \to \infty.$$

This implies $x \in \liminf_n H_n$.

Theorem 2 *Let X be a reflexive Banah space, and let $f_n \in X^* \backslash \{\theta\}$ and $H_n = \{x \in X : f_n(x) = 1\} (n = 0, 1, 2, \cdots)$. Then $\lim_n \|f_n - f_0\| = 0$ if and only if $\theta \notin w - \limsup_n H_n \subset H_0$.*

Proof Suppose $\lim_n \|f_n - f_0\| \neq 0$. Then there exist $\varepsilon_0 > 0$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}_{n=1}^{\infty}$ such that for each $k \in \mathbf{N}, \|f_{n_k} - f_0\| \geq \varepsilon_0$. Since X is reflexive , for each $k \in \mathbf{N}$ there exists $x_k \in S(X)$ such that

$$|f_{n_k}(x_k) - f_0(x_k)| = \|f_{n_k} - f_0\| \geq \varepsilon_0. \tag{3}$$

From $\theta \notin w - \limsup_n H_n$ it is easily seen that $\theta \notin \limsup_n H_n$. By the Lemma we know that $\{f_n\}$ is norm bounded. Thus $\{f_{n_k}(x_k)\}$ is a bounded number sequence and so it has a convergent subsequence. Without loss of generality, we may assume that $f_{n_k}(x_k) \to \alpha$ as $k \to \infty$.

Since X is reflexive $\{x_k\}$ has a weak-convergent subsequence. Without loss of generality we may assume that $x_k \xrightarrow{w} x_0 \in X$.

If $\alpha \neq 0$, we may assume that $f_{n_k}(x_k) \neq 0$ for all $k \in \mathrm{N}$. Now let

$$y_0 = \frac{1}{\alpha}x_0 \quad \text{and} \quad y_k = \frac{1}{f_{n_k}(x_k)}x_k \quad (k = 1, 2, \cdots).$$

Then $y_k \in H_{n_k}(k = 1, 2, \cdots)$ and for every $\phi \in X^*$ we have that

$$\phi(y_k) = \frac{1}{f_{n_k}(x_k)}\phi(x_k) \to \frac{1}{\alpha}\phi(x_0) = \phi(y_0)$$

as $k \to \infty$, which means that $y_k \xrightarrow{w} y_0$ as $k \to \infty$. Thus we obtain that $y_0 \in w - \limsup_n H_n$. By the hypothesis $w - \limsup_n H_n \subset H_0$ we have that $y_0 \in H_0$ and so $f_0(y_0) = 1$. Therefore $f_0(x_0) = \alpha f_0(y_0) = \alpha$. On the other hand, from (3) we obtain that $|\alpha - f_0(x_0)| \geq \varepsilon_0$,a contradiction.

If $\alpha = 0$, *i.e.*, $\lim_k f_{n_k}(x_k) = 0$. From (3) we see that $|f_0(x_0)| \geq \varepsilon_0$. Since $\theta \notin w - \limsup_n H_n$, then $\theta \notin \limsup_n H_n$. Hence $\{f_n\}$, and so $\{f_{n_k}\}$ is norm bounded. By Banach-Alaoglu's theorem we know $\{f_{n_k}\}$ has a weak*-convergent subsequence. Without loss of generality, we may assume that $f_{n_k} \xrightarrow{w^*} f$ as $k \to \infty$. By Theorem 1 and the hypothesis we obtain that $\{x \in X : f(x) = 1\} = H \subset \liminf_k H_{n_k} \subset w - \limsup_n H_n \subset H_0$. Since H and H_0 are both hyperplanes in X , hence $H = H_0$ and so $f = f_0$. Thus $f_{n_k} \xrightarrow{w^*} f_0$, which implies that $K - \lim_k H_{n_k} = H_0$. Let $y_0 = \frac{1}{f_0(x_0)}x_0$, then $y_0 \in H_0$. Moreover we may choose $y_k \in H_{n_k}$, for each $k \in \mathrm{N}$, such that $\|y_k - y_0\| \to 0$ as $k \to \infty$. Next let

$$z_k = \frac{1}{f_{n_k}(y_k - \frac{1}{f_0(x_0)}x_k)}[y_k - \frac{1}{f_0(x_0)}x_k]$$

for each $k \in \mathrm{N}$. Clearly, for each $k \in \mathrm{N}$ $z_k \in H_{n_k}$. Furthermore, for every $\phi \in X^*$ we have

$$\phi(z_k) = \frac{1}{f_{n_k}(y_k) - \frac{1}{f_0(x_0)}f_{n_k}(x_k)}[\phi(y_k) - \frac{1}{f_0(x_0)}\phi(x_k)] \to \phi(y_0) - \frac{1}{f_0(x_0)}\phi(x_0) = 0$$

as $k \to \infty$. This shows that $z_k \xrightarrow{w} \theta$, contrary to the hypothesis $\theta \notin w - \limsup_n H_n$.

Conversely, assume that $\lim_n \|f_n - f_0\| = 0$. Let $x \in w - \limsup_n H_n$. Then there exists a sequence $\{x_k\}, x_k \in H_{n_k}(k = 1, 2, \cdots)$, such that $x_k \xrightarrow{w} x$ as $k \to \infty$. Since a weak convergent sequence is norm bounded and $f_{n_k}(x_k) = 1(k \in \mathrm{N})$, from the following inequality

$$|f_0(x) - 1| \leq |f_0(x) - f_0(x_k)| + \|f_0 - f_{n_k}\|\|x_k\| + |f_{n_k}(x_k) - 1|,$$

where the right side tends to zero as $k \to \infty$, we obtain easily $f_0(x) = 1$ which means that $x \in H_0$. Then $w - \limsup_n H_n \subset H_0$. It is obvious that $\theta \notin w - \limsup_n H_n$.

This completes the proof. □

Observe that the proof of the "only if" part of the above Theorem 2 does not require the assumption on reflexivity of the space X.

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连续线性泛函序列收敛的几何特征

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摘 要: 在本文中，我们证明下述主要结果：(i) 设 X 是赋范线性空间，$f_n \in X^*\backslash\{\theta\}, H_n = \{x \in X : f_n(x) = 1\}, n = 0,1,2,\cdots$，则 $w^* - \lim\limits_n f_n = f_0$ 当且仅当 $H_0 \subset \lim\limits_m \inf H_n$ 且 $\theta \notin \lim\limits_n \sup H_n$; (ii) 当 X 是自反的 Banach 空间时，$\lim\limits_n \|f_n - f_0\| = 0$ 当且仅当 $\theta \notin w - \lim\limits_n \sup H_n \subset H_0$. 并简化了文献 [1] 中的有关结果