Journal of Mathematical Research & Exposition
Vol.21, No.4, 483-490, November, 2001

# Periodic Traveling Wave Solution to a Forced Two-Dimensional Generalized KdV-Burgers Equation *

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**Abstract:** We study the periodic traveling wave solutions to a forced two-dimensional generalized KdV-Burgers equation.Some theorems concerning the boundness,existence and uniqueness of solutions are proved.

**Key words:** KdV-Burgers equation; periodic traveling wave solution; boundness; existence and uniqueness.

**Classification:** AMS(1991) 35Q35/CLC O175.14

**Document code:** A **Article ID:** 1000-341X(2001)04-0483-08

## 0. Introduction

The two-dimensional KdV equation was first derived by Kadomtsev and Petviashvili in 1970[1], and it is also referred to as the KP equation. Periodic traveling wave solutions to the KP and modified KP equation have been investigated by Chen and Wen[2]. Aizicovici and Wen[3] studied the existence and uniqueness of anti-periodic traveling wave solution to a forced generalized KP equation with the aid of monotonicity methods[4,5] and Schauder's fixed point theorem. In [6, 7, 8],a class of KdV-Burgers equation are discussed.

In this paper,we study the periodic traveling wave solutions to a forced generalized KdV-Burgers equation. In section 2, we discuss the boundness of solutions. Section 3 contains main results concerning the existence and uniqueness of solutions.

## 1. Formulation of the problem

We consider the generalized inhomogeneous two-dimensional KdV-Burgerse equation

$$\{u_t + [f(u)]_x + \alpha u_{xx} + \beta u_{xxx}\}_x + \delta u_{yy} + \tilde{g} = 0, \quad x, y \in R, t \geq 0, \tag{1.1}$$

where $f \in C^2(\mathbf{R})$,and $\beta > 0, \delta \neq 0$ and $\alpha$ are given constants, while $\tilde{g}$ denotes a real-valued continuous function of $x, y$ and $t$.

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We are interested in solutions of the form

$$u(x,y,t)=u(z), z=kx+ly-\omega t,$$

where $k>0$, and $l,\omega$ are constants,and we assume that $\tilde{g}(x,y,t)=g(kx+ly-\omega t)$, with $g:\mathbf{R}\to\mathbf{R}$

$$g(z+T)=g(z),\quad \forall z\in R, \tag{1.2}$$

where $T>0$ is a fixed constant.

Straightforward computation shows that (1.1) can be reduced the fourth-order ordinary differential equation

$$\begin{cases} U^{(4)}(z)+bU^{(3)}(z)-cU^{(2)}(z)+r\frac{d^2}{dz^2}f(U(z))+g_1(z)=0, \\ U^{(i)}(0)=U^{(i)}(T),\quad i=0,1,2,3, \end{cases} \tag{1.3}$$

where periodic conditions on $U^{(i)}, i=0,1,2,3$ are imposed, and

$$\begin{aligned} &b=\alpha k^{-1}\beta^{-1}, c=\beta^{-1}k^{-4}(\omega k-\delta l^2), \\ &r=\beta^{-1}k^{-2},\quad g_1(z)=\beta^{-1}k^{-4}g(z). \end{aligned} \tag{1.4}$$

Integrating (1.3) twice,we obtain

$$\begin{aligned} &-U''(z)-bU'(z)+cU(z)+F(U(z))=G(z), z\in\mathbf{R}, \\ &U^{(i)}(0)=U^{(i)}(T),\quad i=0,1, \end{aligned} \tag{1.5}$$

where

$$F(r)=-rf(r),\quad r\in\mathbf{R},$$

and

$$G''(z)=g_1(z), G(z+T)=G(z).$$

For convenience,we consider the following periodic boundary value problem P1:

$$-\ddot{x}-b\dot{x}+cx+F(x)=h(z), x:\mathbf{R}\to\mathbf{R}, z\in\mathbf{R}, \tag{1.6$_1$}$$

$$x^{(i)}(0)=x^{(i)}(T),\quad i=0,1. \tag{1.6$_2$}$$

Without loss of generality,we assume $F(0)=0$ and $x(0)=x(T)=0$.

## 2. Preliminary results

Consider the Sobolev space $W^{1,2}(0,T;R)$ and define the subspace of $W^{1,2}$

$$E=\{x\in W^{1,2}|x^{(i)}(0)=x^{(i)}(T), i=0,1\}.$$

**Lemma 1** *Assume that (1)$F\in C(\mathbf{R}), h\in C(\mathbf{R})$, (2) $\inf_{s\in\mathbf{R}}\frac{F(s)}{s}=\lambda>-\infty$, and (3) $1+\frac{T^2}{2}(c+\lambda)>0$, then the solution of the problem (1.6) is bounded, and the following estimate expressions hold*

$$\|x\|\le k\|G\|, \tag{2.1}$$

$$\|x\| \le KT^{1/2}\|G\|_\infty, \tag{2.2}$$

*where* $K = (1/T^2 + c + \lambda)^{-1} > 0, \|.\|_\infty = \sup_{z\in[0,T]} |h(z)|$.

**Proof** By (1.6), we have

$$\langle -\ddot{x}, x\rangle - b\langle \dot{x}, x\rangle + c\langle x, x\rangle + \langle F(x), x\rangle = \langle h, x\rangle, x \in E,$$

i.e.,

$$\int_0^T [\dot{x}^2 + cx^2 + F(x)x]\mathrm{d}z = \int_0^T h(z)x\mathrm{d}z. \tag{2.3}$$

From Cauchy inequality,we have

$$\int_0^T |x|^2\mathrm{d}z \le \frac{T^2}{2}\int_0^T |\dot{x}|^2\mathrm{d}z. \tag{2.4}$$

From condition (2) and Schwartz inequality,the (2.3) becomes

$$(\frac{2}{T^2} + c + \lambda)\int_0^T |x|^2\mathrm{d}z \le (\int_0^T |h(z)|^2\mathrm{d}z)^{1/2}(\int_0^T |x|^2\mathrm{d}z)^{1/2}.$$

Hence we obtain

$$\|x\| \le (\frac{2}{T^2} + c + \lambda)^{-1}\|h\|. \tag{2.5}$$

On the other hand ,we have

$$(\frac{2}{T^2} + c + \lambda)\int_0^T |x|^2\mathrm{d}z \le \sup_{z\in[0,T]} |h(z)|\int_0^T |x|\mathrm{d}z \le T^{1/2}\|h(z)\|_\infty\|x\|.$$

It then follows that

$$\|x\| \le T^{1/2}(\frac{2}{T^2} + c + \lambda)^{-1}\|h\|_\infty. \tag{2.6}$$

The proof is completed.

**Corollary 1** *Under the conditions of Lemma 1, if* $F(x)$ *is monotonically nondecreasing,* $1 + \frac{T^2}{2}c > 0$, *then the solution is bounded for the problem (1.6),and*

$$\|x\| \le \bar{k}\|h\|, \tag{2.7}$$

$$\|x\| \le T^{1/2}\bar{k}\|h\|_\infty, \tag{2.8}$$

*where* $\bar{k} = (\frac{2}{T^2} + c)^{-1}$.

Since $f(x)$ is continous and monotonically nondecreasing and $f(0) = 0$, thus $xf(x) \ge 0$. Taking $\lambda = 0$ in each expression of Lemma 1,then it the yields the corollary 1.

**Lemma 2** *Under the conditions (1) and (2) of Lemma 1, and* $b \ne 0$, *then the solution is bounded for the problem (1.6), and*

$$\|x\| \le \begin{cases} \frac{T}{\sqrt{2}}|b|^{-1}\|h\|, \\ \|x\| \le \frac{T^{3/2}}{\sqrt{2}}|b|^{-1}\|h\|_\infty. \end{cases} \tag{2.9}$$

**Proof** By the (1.5), making inner product ,we have

$$b\int_0^T |\dot{x}|^2 \mathrm{d}z + \int_0^T F(x)\dot{x}\mathrm{d}z = \int_0^T h(z)\dot{x}\mathrm{d}z, x \in E, \tag{2.10}$$

where, since $x(0) = x(T)$

$$\int_0^T F(x)\dot{x}\mathrm{d}z = \int_{x(0)}^{x(T)} F(x)\mathrm{d}x = 0.$$

Hence

$$|b|\int_0^T |\dot{x}|^2 \mathrm{d}z \le (\int_0^T |h(z)|^2 \mathrm{d}z)^{1/2}(\int_0^T |\dot{x}(z)|^2 \mathrm{d}z)^{1/2},$$

which implies

$$\|\dot{x}\| \le b^{-1}\|h\|. \tag{2.11}$$

It follows from (2.4) that

$$\|x\| \le \frac{T}{\sqrt{2}}|b|^{-1}\|h\|. \tag{2.12}$$

Similarly,we have

$$\|\dot{x}\| \le T^{1/2}|b|^{-1}\|h\|_\infty, \tag{2.13}$$

$$\|x\| \le \frac{T^{3/2}}{\sqrt{2}}|b|^{-1}\|h\|_\infty. \tag{2.14}$$

The proof is complete.

From

$$|x(t)| \le T^{1/2}(\int_0^T |\dot{x}|^2 \mathrm{d}z)^{1/2} \tag{2.15}$$

and Lemmas 1 and Lemma 2, we can obtain following corollaries.

**Corollary 2** *Assume that all conditions of Lemma 2 hold,then the solution of the problem (1.6) is bounded, and*

$$|x(z)| \le \begin{cases} T^{1/2}|b|^{-1}\|h\|, \\ T|b|^{-1}\|h\|_\infty. \end{cases} \tag{2.16}$$

**Corollary 3** *Assume that all conditions of Lemma 1 and Lemma 2 hold,then*

$$\|x(z)\| \le \begin{cases} \min\{k, \frac{T}{\sqrt{2}}|b|^{-1}\}\|h\|, \\ \min\{kT^{1/2}, \frac{T^{3/2}}{\sqrt{2}}|b|^{-1}\}\|h\|_\infty. \end{cases} \tag{2.17}$$

**Corollary 4** *Under the conditions of Lemma 1 or Lemma 2, if $h(z) \equiv 0$, then non-trival periodic solution cannot exist for the problem (1.6).*

## 3. Main theorems

We firstly consider the problem $P_2$:

$$-\ddot{x} + F(x) = h(z), \tag{3.1}$$

$$x^{(i)}(0) = x^{(i)}(T), \quad i = 0, 1, \tag{3.2}$$

and define a functional $J : E \to R$ as

$$J(x) = \int_0^T [\frac{1}{2}|\dot{x}|^2 + V(x)]\mathrm{d}z - \langle h, x\rangle, \forall x \in E, \tag{3.3}$$

where $V(x) = \int_0^x F(x)\mathrm{d}x$. Then it is easy to prove the following variational principle.

**Lemma 3** *The critical points of J in E are the solutions of (3.1) and (3.2).*

**Lemma 4** *On E the usual $W^{1,2}$ norm is equivalent to the following norm:*

$$\|\dot{x}\|_{L^2} = (\int_0^T |\dot{x}|^2\mathrm{d}t)^{1/2}, \forall x \in E. \tag{3.4}$$

**Proof** We know that for all $x \in E$, (2.4)and (2.15) hold, and

$$\int_0^T |x|^2\mathrm{d}z \le T \max_{z\in I} |x(t)|^2 \le T^2 \int_0^T |\dot{x}|^2\mathrm{d}z,$$

$$\int_0^T |\dot{x}|^2\mathrm{d}z \le \int_0^T (|\dot{x}|^2 + |x|^2)\mathrm{d}z \le (1 + \frac{T^2}{2}) \int_0^T |\dot{x}|^2\mathrm{d}z. \tag{3.5}$$

**Theorem 1** *Suppose that (1) $F(x) \in C(\mathbf{R}), h(z) \in [0,T]$, $h(z+T) = h(z)$ and $\int_0^T h(z)\mathrm{d}z = 0$, (2) $V(x) > M, \forall x \in \mathbf{R}$. Then Problem $P_2$ has at least one T-periodic solution.*

**Proof** By (2.4) , we have

$$\langle h, x\rangle \le \|h\|_{L^2}\|x\|_{L^2} \le \frac{T}{\sqrt{2}}\|h\|_{L^2}\|\dot{x}\|_{L^2}. \tag{3.6}$$

From (3.3),(3.6), we obtain

$$J(x) \ge \frac{1}{2}\|\dot{x}\|_{L^2} - \frac{T}{\sqrt{2}}\|h\|_{L^2}\|\dot{x}\|_{L^2} + MT. \tag{3.7}$$

It follows that $J(x_n) \to \infty$ when $\|x_n\|_E = \|\dot{x}_n\|_{L^2} \to \infty$,and that $J(x)$ is bounded from below in $E$. Hence by the standard regularity theory,the minimum solution $x(z)$ is a $C^2$ solution of $P_2$[9,10].

**Theorem 2** *Assume that (1) $F(x) \in C(\mathbf{R}), h(z) \in [0,T], h(z+T) = h(z)$ and $\int_0^T h(z)\mathrm{d}z = 0$,(2)$\inf_{s\in R} \frac{F(s)}{s} = \lambda > -\infty$, and (3) $T(1 + \frac{T^2}{2})^{1/2} < \frac{1}{k}$ as $\lambda \ge 0$ or $(1 + \frac{\lambda T^2}{2})\theta > k$ as $\lambda < 0$, where $K = \max\{|b|, |C|\}, \theta = \sqrt{\frac{\lambda}{\lambda - 1}}$. Then the problem $P_1$ has at least one solution $x \in C^2[0,T]$.*

**Proof** Let $W \in C^2[0,T] \cap E$ is any $T$-periodic function. We consider the periodic problem

$$-x''_w + F(x_w) = h(z) + b\dot{w} - cw, 0 \le z \le T, \tag{3.8}$$

$$x_w^{(i)}(0) = x_w^{(i)}(T), i = 0, 1. \tag{3.9}$$

By Theorem 1, (3.8),(3.9) have at least one $T$-periodic solution $x_w \in C^2(\mathbf{R})$ for a given $w \in E$. Moreover, the map $\Re$, defined by $\Re w = x_w$, is continuous. We can show that $\Re$ is compact in $E \subset W^{1,2}$ as well. Assume that $\|w\|_E \le r$ for some $r > 0$, i.e.,$\|w\|_{c[0,T]} \le r$ or $\|w\|_{L^2} \le r$ by Lemma 4. Taking inner product for (3.8), we obtain

$$\int_0^T |\dot{x}_w|^2 dz + \lambda \int_0^T |x_w|^2 \mathrm{d}z \le \int_0^T (h(z) + b\dot{w} - cw) x_w \mathrm{d}z. \tag{3.10}$$

From conditions (2) and (3) in Theorem ,we have

$$\begin{aligned} \frac{2}{T^2} \int_0^T |x_w|^2 \mathrm{d}z + \lambda \int_0^T |x_w|^2 \mathrm{d}z &\le \int_0^T [|h(z)| + |b\dot{w} - cw|] |x_w| \mathrm{d}z \\ &\le \sup_{z \in I} |h(z)| \int_0^T |x_w| \mathrm{d}z + K \int_0^T (|\dot{w}| + |w|) |x_w| \mathrm{d}z \\ &\le (T^{1/2} \sup_{z \in I} |h(z)| + \sqrt{2} K \|w\|_E) \|x_w\|_{L^2}, \end{aligned} \tag{3.11}$$

where $K = \max\{|b|, |c|\}, I = [0, T]$. Therefore this implies

$$\|x_w\|_{L^2} \le \frac{T^2}{2 + \lambda T^2} (T^{1/2} \|h\|_\infty + \sqrt{2} K r), \tag{3.12}$$

where $\|h\|_\infty = \sup_{z \in I} |h(z)|$. On the other hand,by (3.10), when $\lambda \ge 0$ we have

$$\int_0^T |\dot{x}|^2 \mathrm{d}z \le \{T^{1/2} \|h\|_\infty + \sqrt{2} K \|w\|_E\} \frac{T}{\sqrt{2}} \|\dot{x}_w\|_{L^2},$$

thus

$$\|\dot{x}_w\|_{L^2} \le \frac{T}{\sqrt{2}} \{T^{1/2} \|h\|_\infty + \sqrt{2} K r\}. \tag{3.13}$$

When $\lambda < 0$, we have

$$\|\dot{x}_\omega\|_{L^2} \le \frac{\sqrt{2} T}{2 + \lambda T^2} \left\{T^{1/2} \|h\|_\infty + \sqrt{2} K r\right\}. \tag{3.14}$$

By (3.12) to (3.14),and (2.15),(3.5),they clearly imply that the family $\{x_w\}$ is equicontinuous. Futhermore, from (2.15),(3.13) and (3.14) it follows that $\{x_w\}$ is uniformly bounded. By virtue of Ascoli-Arzela theorem,$\{x_w\}$ is relatively compact in $C[0, T]$,i.e., $\Re$ is compact. Now choose $r > 0$ such that

$$\mu \{T^{1/2} \|h\|_\infty + \sqrt{2} K r\} < r,$$

or equivalently

$$T^{1/2} \|h\|_\infty < \frac{1}{\mu} (1 - \sqrt{2} k \mu) r, \tag{3.15}$$

where $\mu = \sqrt{\frac{1}{2 + T^2}}$ as $\lambda \ge 0$, or $\mu = \frac{\sqrt{2} T}{(2 + \lambda T^2) \theta}$ as $\lambda < 0$ and $\theta = \sqrt{\frac{\lambda}{\lambda - 1}}$. This is clearly possible by condition (3) in Theorem 2 . From Lemma 4,we have $\|x_w\|_E \le r$ and $\|\dot{x}\|_{L^2} \le r$

or $\|x_w\|_{L^2} \le r$ and $|x_w| \le r, \forall z \in I$. This shows that $\Re$ maps a suitable ball $B = B(0,r) \subset C[0,T] \cap E$ into itself, i.e., $\Re B \subset B$. By applying Schauder's fixed point theorem, it follows that $\Re$ has a fixed point $\bar{w}$,i.e., $\bar{w} = x_w$.Consequently,$x = x_{\bar{w}} \in C^2[0,T]$ is the desired solution of the problem $P_1$, since both $h$ and $F$ are continuous.

The proof is complete.

**Theorem 3** *Assume that (1)* $h(z) \in [0,T], h(z+T) = h(z), \int_0^T h(z)dz = 0$, *(2)* $F(x)$ *is continuous and monotonically nondecreasing, (3)* $1 - KT(1+\frac{T^2}{2})^{1/2} > 0, K = \max\{|b|,|c|\}$, *then the problem* $P_1$ *has at least one solution* $x \in C^2[0,T]$.

**Proof** From the condition (2) and $F(0) = 0$,we have $xf(x) \ge 0$. Taking $\lambda = 0$ in the proof of Theorem 2,then the theorem is proved.

**Theorem 4** *Assume that (1) the problem* $P_1$ *has a T-periodic solution,(2)* $F(x)$ *is Lipschitz continuous,(3)* $1 + \frac{T^2}{2}(c - L) > 0$, *where* $L > 0$ *is the Lipschitz contstant,then the problem* $P_1$ *has a unique solution* $x \in C^2(\mathbf{R})$.

**Proof** Suppose that $x_i (i = 1,2)$ are two different solutions of the problem $P_1$. Let $v = x_1 - x_2$, then

$$-v'' - bv' + cv + F(x_1) - F(x_2) = 0, \tag{3.16}$$

$$v^{(i)}(0) = v^{(i)}(T), i = 1,2.$$

Taking inner product for equation (3.16), we have

$$\int_0^T |v'|^2 dz + c\int_0^T v^2 dz + \int_0^T (F(x_1) - F(x_2))v dz = 0. \tag{3.17}$$

Hence

$$|\int_0^T |v'|^2 dz + c\int_0^T v^2 dz| \le L\int_0^T v^2 dz. \tag{3.18}$$

From (2.4) and the condition (3), it yields

$$(\frac{2}{T^2} + c - L)\|v\|_{L^2}^2 \le 0. \tag{3.19}$$

Hence

$$\|v(z)\|_{L^2} = 0, \quad \forall z \in \mathbf{R}. \tag{3.20}$$

By (3.18),we have $\|v'\|_{L^2} = 0$. Futhermore, it follows from that (2.15) $|v| = 0$,i.e., $x_1 = x_2, \forall z \in \mathbf{R}$. This implies the problem $P_1$ has a unique solution.

**Theorem 5** *Assume that (1) the condition (1) of Theorem 4 holds, (2)* $F(x)$ *is continuous and monotonically nondecreasing ,(3)* $\frac{2}{T^2} + c > 0$, *then the problem* $P_1$ *has at most one solution.*

**Proof** From condition (2) ,we have

$$\int_0^T |v'|^2 dz + c\int_0^T v^2 dz \le 0. \tag{3.21}$$

Therefore

$$(\frac{2}{T^2}+c)\|v\|_{L^2}^2 \le 0. \tag{3.22}$$

By $\frac{2}{T^2}+c>0$. We have $\|v\|_{L^2}^2=0$. Similarly,we have $|v|=0$,i.e.,$x_1=x_2$, The theorem is proved.

## References:


[1] KADOMTSEV B B, PETVIASHVILI V I. *On the stability of solitary waves in weakly dispersing media* [J]. Soviet Phys Dokl, 1970, **15**: 539–541.

[2] CHEN Y, WEN S. *On existence theorems of periodic traveling wave solutions to the two-dimensional Korteweg-deVries equation* [J]. Appl. Anal., 1988, **31**: 1–10.

[3] AIZICOVICI S, WEN S. *Anti-periodic traveling wave solution to a forced two- dimensional generalized Korteweg-deVries equation* [J]. J.Math. Anal. Appl., 1992, **174**: 556–565.

[4] AFTABIZADEH A R, AIZICOVICI S, PAVEL N H. *On a class of second-order anti-periodic boundary value problems* [J]. J. Math. Anal. Appl., 1992, **171**: 301–320.

[5] AIZICOVICI S, PAVEL N H. *Anti-periodic solutions to a class of nonlinear differential equation in Hilbert space* [J]. J. Funct. Anal., 1991, **99**: 387–408.

[6] AMIK C J, BONA J L, SCHONBEK M E. *Decay of some nonlinear wave equations* [J]. J. Diff. Eqs., 1989, **81**(1): 1–49.

[7] BONA J L, DOUGALIS V A, KARADASHIAN O A, et al. *Computation of blow-up and decay for periodic solutions of the generalized Korteweg-deVries-Burgers equations* [J]. Appl. Numerical Math., 1992, **10**(3-4): 335–355.

[8] TAN J Y. *Periodic traveling wave solutions to a forced generalized Korteweg-deVries-Burgers equation* [J]. J. Chongqing University, 1996, **19**(6): 90–95.

[9] MAWHIN J, WILLEM M. *Critical Point Theory and Hamiltonian Systems* [M]. New York: Springer-Verleg, 1989, 1-274.

[10] RABINOWITZ P H. *Minimax methods in critical point theory with applications to differential equations* [J]. CBMSReg Conf Ser in Math., 1986, **65**: 1–10.


# 受迫二维广义 KdV-Burgers 方程的周期行波解


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**摘 要**: 本文研究了受迫二维广义 KdV-Burgers 方程的周期行波解问题，讨论了解的有界性并给出了解的估计式，进而讨论了周期解的存在性及唯一性.