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On the Linear Scheme for the Reissner-Mindlin Plate Problem *

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Abstract: In this paper we give the optimal selection of the bubble function in the linear scheme proposed by recent paper [1] for the Reissner-Mindlin plate problem.

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In this paper we discuss the optimal selection of the bubble function in the linear scheme proposed by X.L. Cheng, W. Han and H.C. Huang in [1] for the Reissner-Mindlin plate problem.

Let Ω denote the region in R^2 occupied by the midsection of the plate, and denote by w and $\vec{\varphi}$ the transverse displacement of Ω and the rotation of the fibers normal to Ω, respectively. The Reissner-Mindlin plate model determines $(w,\vec{\varphi})$ as the unique solution to the following variational problem: Find $(w,\vec{\varphi}) \in H_0^1(\Omega)\times[H_0^1(\Omega)]^2$ such that

$$a(\vec{\varphi},\vec{\psi}) + \lambda t^{-2}(\vec{\varphi}-\nabla w, \vec{\psi}-\nabla\mu) = (g,\mu), \quad \forall\,(\mu,\vec{\psi}) \in H_0^1(\Omega)\times[H_0^1(\Omega)]^2. \qquad (1)$$

Here g is the scaled transverse loading function, t is the plate thickness, $\lambda = Ek/2(1+\nu)$ with E the Young's modulus, ν the Poisson ratio, k the shear correction factor, and the parentheses denote the usual L^2 inner product. The bilinear form $a(\cdot,\cdot)$ is defined as

$$a(\vec{\varphi},\vec{\psi}) = \frac{E}{12(1-\nu^2)}\int_\Omega[(\frac{\partial\varphi_1}{\partial x}+\nu\frac{\partial\varphi_2}{\partial y})\frac{\partial\psi_1}{\partial x}+(\nu\frac{\partial\varphi_1}{\partial x}+\frac{\partial\varphi_2}{\partial y})\frac{\partial\psi_2}{\partial y}+ \frac{1-\nu}{2}(\frac{\partial\varphi_1}{\partial y}+\frac{\partial\varphi_2}{\partial x})(\frac{\partial\psi_1}{\partial y}+\frac{\partial\psi_2}{\partial x})]\mathrm{d}x\mathrm{d}y, \qquad (2)$$

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where φ_1, φ_2 and ψ_1, ψ_2 are the components of $\vec{\varphi}$ and $\vec{\psi}$. It can be proved, by using Korn's inequality, that $a(\cdot,\cdot)$ is an inner product on $[H_0^1(\Omega)]^2$ equivalent to the usual one. For simplicity, we will assume

$$a(\vec{\varphi},\vec{\psi}) = (\nabla\vec{\varphi},\nabla\vec{\psi}). \tag{3}$$

Let Ω be a convex polygon and $\Im_h$ be a regular triangular partition of Ω where as usual h stands for the mesh size. Define finite element spaces by

$$W_h = \{v \in H_0^1(\Omega) : v|_T \in P_1(T),\ \forall T \in \Im_h\}, \tag{4}$$

$$B_h = \{v \in H_0^1(\Omega) : v|_T \in \text{span}\{b_T\},\ \forall T \in \Im_h\}, \tag{5}$$

where b_T is a bubble function on element T. A natural way is to choose $b_T = \lambda_1\lambda_2\lambda_3$, where $\lambda_i (i = 1,2,3)$ are the barycentric coordinates in triangle T. We can also choose a hat-function for b_T. Denote

$$H_0(\text{rot};\Omega) = \{\vec{\mu} \in [L^2(\Omega)]^2 : \text{rot}\vec{\mu} \in L^2(\Omega), \vec{\mu}\cdot\vec{\tau} = 0 \text{ on } \partial\Omega\}.$$

We define

$$H_h = [W_h \oplus B_h]^2, \tag{6}$$

$$\Gamma_h = \{\vec{\mu} \in [L^2(\Omega)]^2 : \vec{\mu}|_T \in [P_0(T)]^2,\ \forall T \in \Im_h\}, \tag{7}$$

$$Q_h = \{\vec{\mu} \in H_0(\text{rot};\Omega) : \vec{\mu}|_T \in [P_0(T)]^2 \oplus (y,-x)P_0(T),\ \forall T \in \Im_h\}. \tag{8}$$

Define an operator $\mathcal{R}_h : H_0(\text{rot};\Omega) \to Q_h$ by the conditions

$$\int_e (\mathcal{R}_h\vec{s})\cdot\vec{\tau} = \int_e \vec{s}\cdot\vec{\tau}, \quad \forall e \in \partial T,\ \forall T \in \Im_h, \tag{9}$$

where $\vec{\tau}$ is tangential unit vector of the edge e of an element T. It is a reduction operator to the lowest order rotated Raviart-Thomas space Q_h. Also define $\mathcal{P}_h : [L^2(\Omega)]^2 \to \Gamma_h$ by

$$(\mathcal{P}_h\vec{s})|_T = \frac{1}{\text{meas}(T)}\int_T \vec{s}\,\text{d}x\text{d}y. \tag{10}$$

We now define an operator $\pi_h : H_h \to \Gamma_h$. For $\vec{\varphi} \in H_h$, with $\vec{\varphi} = \vec{\varphi}^L + \vec{\varphi}^B$, $\vec{\varphi}^L \in [W_h]^2$ and $\vec{\varphi}^B \in [B_h]^2$, let

$$\pi_h\vec{\varphi} = \mathcal{P}_h\mathcal{R}_h\vec{\varphi}^L + \mathcal{P}_h\vec{\varphi}^B. \tag{11}$$

The linear scheme proposed in [1] was given in the following problem:

Find $(w_h,\vec{\varphi}_h) \in W_h \times H_h$ such that

$$a(\vec{\varphi}_h,\vec{\psi}_h) + \lambda t^{-2}(\pi_h\vec{\varphi}_h - \nabla w_h, \pi_h\vec{\psi}_h - \nabla v_h) = (g,v_h),\ \forall(v_h,\vec{\psi}_h) \in W_h \times H_h. \tag{12}$$

The existence and uniqueness of the solution follow from the coerciveness of $a(\cdot,\cdot)$. Eliminating the bubble function b_T at element level, we can obtain the following problem:

Find $(w_h, \vec{\varphi}_h) \in W_h \times [W_h]^2$ such that

$$a(\vec{\varphi}_h, \vec{\psi}_h) + \sum_{T \in \Im_h} \lambda(t^2 + J(b_T))^{-1}(\pi_h \vec{\varphi}_h - \nabla w_h, \pi_h \vec{\psi}_h - \nabla v_h)_T$$
$$= (g, v_h), \forall (v_h, \vec{\psi}_h) \in W_h \times [W_h]^2, \tag{13}$$

where

$$J(b_T) = (\frac{1}{\text{meas}(T)} \int_T b_T \, dx)^2 / \int_T \nabla b_T \cdot \nabla b_T \, dx. \tag{14}$$

Theorem 1 *Assume $g \in L^2(\Omega)$. Let $(w, \vec{\varphi})$ and $(w_h, \vec{\varphi}_h)$ be the solutions of (1) and (12). Then*

$$\|\vec{\varphi} - \vec{\varphi}_h\|_1 + \|w - w_h\|_1 \le C_1 \, h \, \|g\|_0, \tag{15}$$

where

$$C_1 \le C_2 + \sup_{T \in \Im_h} J(b_T)^{-1}, \tag{16}$$

and C_2 is a constant independent of h and t.

Proof Similar to the proof of Theorem 4.5 (pp.230-231) in [1].

Then there are two criteria to select a good bubble function. From (16), it is indicated that the bigger $J(b_T)$, the smaller constants in the error estimates. From (13), it follows indicating that the bigger $J(b_T)$, the less possibility of the occurence of the locking by the small thickness t. These two criteria imply to determine the optimal bubble function by

$$J(b_T^o) = \sup_{b_T \in H_0^1(T)} J(b_T). \tag{17}$$

Then we can obtain the result using the method in [2]:

Theorem 2 *Let $\tilde{b}_T$ be the solution of*

$$\begin{cases} -\Delta \tilde{b}_T = 1, & in \quad T, \\ \tilde{b}_T = 0, & on \quad \partial T. \end{cases} \tag{18}$$

Then we have

$$J(b_T^o) = \sup_{b_T \in H_0^1(T)} J(b_T) = \frac{1}{\text{meas}(T)} \int_T \tilde{b}_T \, dx. \tag{19}$$

Remark The equivalence of (12) and (13) is based on the assumption of bilinear form (3). Otherwise we obtain the following equivalent formulation:

Find $(w_h, \vec{\varphi}_h) \in W_h \times [W_h]^2$ such that

$$a(\vec{\varphi}_h, \vec{\psi}_h) + \sum_{T \in \Im_h} \lambda(t^2 I_2 + B_T)^{-1}(\pi_h \vec{\varphi}_h - \nabla w_h, \pi_h \vec{\psi}_h - \nabla v_h)_T$$
$$= (g, v_h) \forall (v_h, \vec{\psi}_h) \in W_h \times [W_h]^2, \tag{20}$$

where

$$B_T = \text{means}(T) A_T^{-1}, \qquad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{21}$$

$$A_T = \begin{pmatrix} a((b_T,0),(b_T,0)) & a((0,b_T),(b_T,0)) \\ a((b_T,0),(0,b_T)) & a((0,b_T),(0,b_T)) \end{pmatrix}. \tag{22}$$

The matrix A_T is not diagonal, neither is the matrix B_T. But the error estimates (15) and (16) in Theorem 1 hold without the assumption (3). So the optimal selection of bubble function is the same as Theorem 2.

References:

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[2] PIERRE R. *Optimal selection of the bubble function in the stabilization of the P1-P1 element for the Stokes problem* [J]. SIAM J. Numer. Anal., 1995, **32**: 1210–1224.

关于 Reissner-Mindlin 板问题的线性格式

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摘 要: 本文给出 Reissner-Mindlin 板问题的线性格式 [1] 中的汽泡函数的最优选取.