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Some Planar Graphs with Star Chromatic Number Between Three and Four *

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Abstract: We construct some infinite families of planar graphs with star chromatic number $3+1/d$, $3+2/(2d-1)$, $3+3/(3d-1)$, and $3+3/(3d-2)$, where $d \geq 2$, partially answering a question of Vince.

Key words: (k,d)-coloring; star chromatic number; planar graph.

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1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The graph is simple if there is neither a parallel edge nor a loop. In this paper, we only consider simple graphs. A k-coloring of G is a mapping of assigning the k colors to the vertices such that adjacent vertices receive distinct colors. G is said to be k-chromatic if k is the least integer for which G has a k-coloring and $\chi(G) = k$ is called the chromatic number of G. The star chromatic number [1] of a graph G is denoted by $\chi_c(G)$ which is a natural generalization of the notion of the chromatic number. In such a coloring, one is permitted to use more than $\chi(G)$ colors but the colors assigned to the adjacent vertices should be as far as possible in some sense.

Let x be an integer, $k > 1$ be a positive integer, and $\mathcal{Z}_k = \{0, 1, 2, ..., k-1\}$. Let $|x|_k$ be the distance from x to the nearest multiple of k, d be a positive integer such that $2d < k$ and be coprime with k. A (k,d)-*coloring* of a graph G is a function $c : V(G) \to \mathcal{Z}_k$ such that for any edge uv of $E(G)$, $|c(u) - c(v)|_k \geq d$. Vince has defined the star chromatic number of G as: $\chi_c(G) = inf\ \{k/d :\ G \text{ has a } (k,d)\text{-coloring}\}$.

In [1], Vince raised some open questions. Here are two of them:

(1) What are some infinite family of planar graphs with star chromatic numbers between two and three besides odd cycles?

(2) What are some infinite family of planar graphs with star chromatic numbers between three and four?

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This paper is motivated by question (2). We know little about the existence of planar graphs with star chromatic number between three and four. Till now, we only know the exact value of star chromatic numbers of a few infinite families of planar graphs being between three and four such as triangular prisms, reels and Hajos-sum wheels etc., for details, see[2–8]. Here we provide a few infinite families of planar graphs with star chromatic number between three and four. Obviously, this answers one question of Vince in part.

2. Planar graphs with star chromatic number $3 + 1/d$

Let $P_n^i = (u_1^i u_2^i ..., u_n^i)$ be an $n-1$-path. Let T_n^i be the graph obtained from P_n^i by joining u_j^i to u_{j+2}^i, where $j = 1, 2, ..., n-2$, $(n \geq 3)$. If $i = 0$, the upper-scripts are omitted. Add an isolated vertex u to P_n, let G_1 be the graph obtained from P_n by joining u to u_1 and u_n to u_{3i+2}, where i is one elements of the set $\{0, 1, ..., \lfloor n/3 \rfloor\}$. The valence of u is three in G. It is easily seen that $\chi(G) = \chi_c(G) = 3$ if $|V(G)| = 3k$ or $3k+2$, $\chi(G) = 4$ if $|V(G)| = 3k+1$, where k is a positive integer.

Theorem 2.1 *Let d be a positive integer. $\chi_c(G)$ is $3 + 1/d$ if the order of G is $3d + 1$.*

Proof We give a $(3d+1, d)$-coloring c of G as follows. $c(u) = 0$, $c(u_i) = id \pmod{3d+1}$ where $i = 1, 2, ..., 3d+1$. It is easy to check that it is a legal $(3d+1, d)$-coloring. Next, there is no coprime integer pairs k' and d' such that $3/1 < k'/d' < 3 + 1/d$, $k' \leq 3d+1$, $0 < k' - 3d' < d'/d \leq 1$.

3. Planar graphs with star chromatic number $3 + 2/(2d - 1)$

Let c be a (k, d)-coloring of G. We define a directed graph $D_c(G)$ from G by orienting an edge xy from x to y if $c(y) = c(x) + d \pmod{k}$.

Lemma 3.1[4] *Let G be a connected graph. $\chi(G) = k/d$ if and only if G is (k, d)-colorable, and for any (k, d)-coloring c, $D_c(G)$ contains at least one directed cycle.*

Lemma 3.2 *Let G be a connected graph with a (k, d)-coloring. If there is a color i not appearing in the (k, d)-coloring c, then there is another coprime integer pairs k', d' such that $k'/d' < k/d$ and G is also (k', d')-colorable.*

Proof Since $gcd(k, d) = 1$, there is a integer j such that $jd = i \pmod{k}$, where $0 \leq j \leq k-1$. Therefore, for any (k, d)-coloring of G, $D_c(G)$ will never contain any directed cycle. By Lemma 3.1, $\chi_c(G) = k'/d' < k/d$.

Theorem 3.3 *G is colored uniquely for any $(3d+1, d)$-coloring up to permutations.*

Proof For any $(3d+1, d)$-coloring c, $D_c(G)$ contains a directed hamilton cycle.

We construct a graph G_2 as follows. Take a copy of T_{3d}^1, $(d \geq 2)$ and two isolated vertices x and y. If d is even, join x to u_1^1, u_2^1, u_{3d}^1, and y to u_1^1, u_{3d-1}^1, u_{3d}^1. Take another copy of T_{3d-3}^2, join x to u_1^2, u_2^2, and y to u_{3d-3}^2, if d is odd, join x to u_1^1, u_5^1, u_{3d}^1, and y to u_1^1, u_{3d-1}^1, u_{3d}^1. Take another copy of T_{3d-3}^2, join x to u_1^2, u_2^2, and y to u_{3d-3}^2. G_2 is planar, its order is $6d - 1$.

Theorem 3.4 *$\chi_c(G)$ is $3+2/(2d-1)$.*

Proof We prove that G_2 is not $(3d+1,d)$-colorable. By symmetry, x and T^1_{3d} induce G_1 as a subgraph of G_2. For any $(3d+1,d)$-coloring of G_1. Without loss of generality, assume $c(x)=0$. y and T^1_{3d} induce G_1. By the colors of T^1_{3d}, we get the color of y is also 0, identify x and y, the new vertex x and T^2_{3d-3} induce also a graph G_1 but its order is $1+3(d-1)$. By Theorem 3.2, its star chromatic number is $3+1/d-1$ greater than $3+1/d$. It is unlikely that x, y and T^2_{3d-3} have a legal $(3d+1,d)$-coloring.

Further, we show that G_2 is $(6d-1,2d-1)$-colorable. Define such a $(6d-1,2d-1)$-coloring as follows: $c(x)=0$, $c(u^1_i)=i(2d-1) \pmod{6d-1}$, $i=1,2,..3d$, $c(y)=(3d+1)(2d-1)$, $c(u^2_{3(d-1)})=(3d+2)(2d-1) \pmod{6d-1}$, $c(u^2_i)=(3(d-1)+1-i)(2d-1)+(3d+1)(2d-1) \pmod{6d-1}$. It is easy to check that it is legal.

Finally, we prove that there are no coprime integers k' and d' such that $3+1/d<k'/d'<3+2/(2d-1)$ and $k'\le 6d-1$. Then we have $d'/d<k'-3d'<2d'/(2d-1)$, by $d\le d'\le 2d-1$ then $1\le d'/d<k'-3d'<2d'/(2d-1)\le 2$, a contradiction. Then $\chi_c(G_2)=3+2/(2d-1)$.

4. Planar graphs with star chromatic number $3+3/(3d-1)$

We construct G_3 as follows: take three copies of T_{3d}, denoted by T^1_{3d}, T^2_{3d}, T^3_{3d}. If d is even, join vertex u^2_1 to u^1_1, u^1_2 and u^1_{3d}, u^3_1 to u^2_1, u^2_2 and u^2_{3d}. Join u^1_{3d} to u^2_{3d-1} and u^2_{3d}. u^2_{3d} to u^3_{3d-1} and u^3_{3d}, and join u^3_{3d} to u^1_{3d}, u^1_{3d-1} and u^1_1. If d is odd, join u^2_1 to u^1_1, u^1_2 and u^1_{3d}, u^3_1 to u^2_1, u^2_2 and u^2_{3d}. Join u^1_{3d} to u^2_{3d-4} and u^2_{3d}, u^2_{3d} to u^3_{3d-4} and u^3_{3d}. Join u^3_{3d} to u^1_{3d}, u^1_{3d-4}, and u^1_1.

Lemma 4.1 *G_3 is not $(3d+1,d)$-colorable.*

Proof By contradiction, suppose G_3 is $(3d+1,d)$-colorable. u^2_1 and T^1_{3d}, u^3_1 and T^2_{3d}, u^1_{3d} and T^2_{3d}, u^2_{3d} and T^3_{3d} induce a graph G_1. For any $(3d+1,d)$-coloring of G_1, the colors of u^1_1 and T_{3d} will induce a $(3d+1,d)$-coloring of G_1, without loss of generality, we assume that $c(u^2_1)=0$, since $\chi_c(G_1)=3+1/d$, and any $(3d+1,d)$-coloring of G_1 is unique if the colors of one more vertex of G_1 is known. There are only two cases to consider.

Case 1 $c(u^1_1)=d$, then $c(u^1_{3d})=3d^2 \pmod{3d+1}=-d=2d+1$, u^1_{3d} and T^2_{3d} induce also a graph of G_1, the colors of u^2_1 and u^1_{3d} are known, the colors of T^2_{3d} can be determined. $c(u^2_{3d})=(3d-1)d \pmod{3d+1}=d+1$. By the same reason, $c(u^3_1)=2d+1$, and $c(u^3_{3d})=1$. but $|c(u^3_{3d})-c(u^1_1)|=d-1$.

Case 2 If $c(u^1_{3d})=d$, then $c(u^1_1)=2d+1$, $c(u^2_{3d})=2d$, $c(u^3_1)=d$ and $c(u^3_{3d})=3d$, and also we have $c(u^3_{3d})-c(u^1_1)=3d-(2d-1)=1$.

Then G_3 is not $(3d+1,d)$-colorable.

Lemma 4.2 *G_3 is not $(9d,3d-1)$-colorable.*

Proof We define such a $(9d,3d-1)$-coloring as follows. $c(u^1_1)=0$, $c(u^1_i)=(i-1)(3d-1) \pmod{9d}$, $i=1,2,...,3d$. $c(u^1_1)=0$, $c(u^1_2)=(3d-1)$, $c(u^1_{3d-1})=2$. $c(u^1_{3d})=3d+1$, $c(u^2_i)=3d+1+i(3d-1)$, $i=1,2,...,3d$. $c(u^2_i)=6d$, $c(u^2_2)=9d-1$, $c(u^2_{3d-1})=6d+2$,

$c(u_{3d}^2) = 1$. $c(u_i^3) = 3d + i(3d-1) \pmod{9d}$, then $c(u_{3d}^3) = 6d+1$. Then $\chi_c(G_3) \le 3 + 3/(3d-1)$.

Theorem 4.3 *$\chi_c(G_3)$ is $3 + 3/(3d-1)$.*

Proof By Lemmas 3.1 and 3.2, we know that $3 + 1/d < \chi_c \le 3 + 3/(3d-1)$. It suffices to prove that there are not coprime integers k' and d' such that $3 + 1/d < k'/d' < 3 + 3/(3d-1)$ and $k' \le 9d$. By the above inequalities, $d'/d < k' - 3d' < 3d'/(3d-1)$ and $d \le d' \le 3d-1$, then $1 \le d'/d < k' - 3d' < 3d'/(3d-1) \le 3$. The possible case is $k' = 3d' + 2$. If it were, then $1/d < 2/d' < 3/(3d-1)$, and $2d - 2/3 < d' < 2d$, it is clearly false. Then, $\chi_c(G_3) = 3 + 3/(3d-1)$.

5. Planar graphs with star chromatic number $3 + 3/(3d-2)$

We construct an infinite family of planar graphs G_4 from G_2 as follows: take two copies of T_{3d}, denoted by T_{3d}^1, T_{3d}^2, and one copy of $T_{3(d-1)}$, denoted by $T_{3(d-1)}^3$, and one isolated vertex u. If d is even, join u to u_1^1, u_{3d-1}^1, and u_{3d}^1, u_{3d-3}^3, u_{3d-4}^3, u_1^2 to u_1^1, u_2^1, u_{3d}^1, and u_1^3, u_{3d}^1 to u_{3d}^2 and u_{3d-1}^2, and identify u_1^3 and u_{3d}^2; If d is odd, join u to u_1^1, u_{3d}^1, u_{3d-1}^1, u_{3d-3}^3, and u_{3d-4}^3, u_1^2 to u_1^1, u_5^1, u_{3d}^1 and u_1^3, u_{3d}^1 to u_{3d}^2 and u_{3d-1}^2, and identify u_1^3 and u_{3d}^2.

Lemma 5.1 *G_4 is not $(6d-1, 2d-1)$-colorable.*

Proof On the contrary, if it were, for any $(6d-1, 2d-1)$ -coloring of G_4, u, u_1^2, T_{3d}^1 and T_{3d-3}^3 induce a graph G_2 of star chromatic number $3 + 2/(2d-1)$, the color of u, u_1^2, T_{3d}^1 and T_{3d-3}^3 will induce a $(6d-1, 2d-1)$-coloring of G_2. It is unique to color G_2 with any $6d-1, 2d-1)$-coloring up to permutation. If two colors of the adjacent vertices are known, the colors of all other vertices of G_2 can be determined. Without loss of generality, $c(u_i^1) = 0$, $c(u_i^1) = i(2d-1) \pmod{6d-1}$, $i = 1, 2, ..3d$, $c(u_{3d}^1) = 4d - 1$, $c(u_1^1) = 2d - 1$, $c(u_2^1) = 4d - 2$, $c(u_1^2) = 6d - 2$, $c(u_i^2) = 4d - 1 + i(2d-1) \bmod (6d-1)$ $i = 1, 2, .., 3d$, $c(u_{3d}^2) = 2d - 1$. By the colors of u_{3d}^1 and u_{3d}^2 the color of u_{3d-1}^2 is determined uniquely. It is $6d - 2$, identify u_1^2 and u_{3d-1}^2, u_1^2, u_2^2, .., u_{3d-1}^2 induce a graph of star chromatic number not less than $3 + 1/(d-1)$. Since the induced graph contains G_1 as its subgraph, $\chi_c(G_1) = 3 + 2/(2d-1)$. G_1 cannot have a $(6d-1, 2d-1)$-coloring, a contradiction. Thus G_4 is not $(6d-1, 2d-1)$-colorable.

Lemma 5.2 *G is $(9d-3, 3d-2)$-colorable.*

Proof Define such a $9d-3, 3d-2)$-coloring c as follows: $c(u) = 0$, $c(u_i^1) = i(3d-2) \pmod{9d-3}$, $i = 1, 2, ..., 3d$. We can check that it is a legal (k, d)-coloring.

Theorem 5.3 *$\chi_c(G_4) = 3 + 3/(3d-2)$.*

Proof By Lemmas 5.1 and 5.2, $3 + 2/(2d-1) < \chi_c \le 3 + 3/(3d-2)$. There is no k' and d' such that $k' \le 9d - 3$, since $2 \le 3d'/(2d-1) < k' - 3d' < 3d'/(3d-2) \le 3$, and so $\chi_c = 3 + 3/(3d-2)$.

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星色数在三与四之间的一些平面图类

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摘 要: 本文构造出了星色数为 $3+1/d$, $3+2/(2d-1)$, $3+3/(3d-1)$, 和 $3+3/(3d-2)$ 的一些平面图类，从而部分解决了 Vince 的问题.