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Asymptotic Expansion of Some Sheffer Polynomials *

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Abstract: Asymptotic expansion of two Sheffer polynomials, namely, Charlier and Laguerre, was obtained by L.C. Hsu[6] using a combinatorial method. In this paper, L.C. Hsu's method in [6] has been put into a formal theorem that the author successfully applied to four other Sheffer polynomials: Poisson-Charlier, weighted Touchard, Toscano, and Angelescu polynomials. Within some specified domains remainder estimates have been obtained. Moreover, some spplicability and limitation have been mentioned.

Key words: asymptotic expansion; Sheffer polynomials.

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1. Introduction

Sheffer polynomials are generated by functions of the form (cf. Boas and Buck[1]

$$A(t)e^{zg(t)} = \sum_{n=0}^{\infty} p_n(z)t^n,$$

where $A(t)$ and $g(t)$ are functions analytic on some domain containing zero, with $A(0) = 1, g(0) = 0$ and $g'(0) \neq 0$.

The importance of these polynomials lies in their being the coefficients of power series expansion of analytic functions. Roman and Rota[14] treated these type of polynomials using the method of umbral calculus. L.C. Hsu and Peter Shiue[7] applied the cycle indicator method to some of these polynomials and come up with a list of such polynomials which are C_n-representable.

In this paper we obtain asymptotic expansion of the following Sheffer polynomials: Poisson-Charlier, weighted Touchard, Toscano, and Angelescu polynomials, as a parameter λ goes to positive infinity under some restrictions with respect to the degree of the given

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polynomial. We will use the cycle indicator method that was first used by L.C. Hsu[6], although he did not call it cycle-indicator method. Here L.C. Hsu's method has been put into a formal theorem and is successfully applied to the four polynomials mentioned.

2. The cycle indicator method

The cycle indicator C_n may be written (cf. Riordan[11])

$$C_n(t_1, t_2, \cdots, t_n) = \sum_{\sigma(n)} \frac{n!}{k_1!k_2!\cdots k_n!}\left(\frac{t_1}{1}\right)^{k_1}\left(\frac{t_2}{2}\right)^{k_2}\cdots\left(\frac{t_n}{n}\right)^{k_n}, \tag{2.1}$$

where the sum is over all non-negative integral values of k_1 to k_n such that $k_1 + 2k_2 \cdots + nk_n = n$, or what is the same thing, over the set $\sigma(n)$ of all partitions of n.

Suppose that a polynomial $p_n(z)$ can be written in the form

$$p_n(z) = \mu C_n(f_1, f_2, \cdots, f_n), \tag{2.2}$$

where the f_i's are functions of z and μ is some constant. We call (2.2) the C_n-representation of $p_n(z)$. The concept of C_n-representation first began when several authors like Gessel, Konvalina and MacMahon (as mentioned in [7]) expressed some polynomials and number sequences in the form of the cycle indicator C_n of the symmetric group.

The cycle indicator method of finding asymptotic expansion of some polynomial sequences uses the C_n-representation of such polynomials.

Suppose that we wish to find an asymptotic expansion of the polynomial $p_n(\lambda z)$ as $n \to \infty, \lambda \to \infty$ such that $n = o(\lambda^{1/2})$.

Assume that $p_n(\lambda z)$ can be written in the form $p_n(\lambda z) = \frac{1}{n!}C_n(f_1, f_2, \cdots, f_n)$, where $f_i = a_i + b_i\lambda z$ for each $i, i = 1, 2, 3, \cdots, n$, and a_i, b_i are bounded coefficients. Then it follows from (2.1) that

$$\begin{aligned} p_n(\lambda z) &= \sum_{\sigma(n)} \frac{(a_1 + b_1\lambda z)^{k_1}(\frac{a_2}{2} + \frac{b_2}{2}\lambda z)^{k_2}\cdots(\frac{a_n}{n} + \frac{b_n}{n}\lambda z)^{k_n}}{k_1!k_2!\cdots k_n!} \\ &= \sum_{j=0}^{n-1}\sum_{\sigma(n,n-j)} \frac{(a_1 + b_1\lambda z)^{k_1}(\frac{a_2}{2} + \frac{b_2}{2}\lambda z)^{k_2}\cdots(\frac{a_n}{n} + \frac{b_n}{n}\lambda z)^{k_n}}{k_1!k_2!\cdots k_n!} \\ &= \sum_{j=0}^{n-1}\lambda^{n-j}\sum_{\sigma(n,n-j)}\prod_{i=1}^{n}\frac{1}{k_i!}\left(\frac{a_i}{i\lambda} + \frac{b_i}{i}z\right)^{k_i}, \end{aligned}$$

with λ a large real parameter. The second equality uses the fact that $\sigma(n) = \cup_{k=1}^{n}\sigma(n,k)$ where $\sigma(n,k)$ denote the set of partitions of n with number of parts equal to k, i.e.,

$$\sigma(n,k) = \{1^{k_1}2^{k_2}\cdots n^{k_n} : k_1 + 2k_2 + \cdots + nk_n = n; k_1 + k_2 + \cdots + k_n = k\}.$$

Letting $U_j = \sum_{\sigma(n,n-j)}\prod_{i=1}^{n}\frac{1}{k_i!}(\frac{a_i}{i\lambda} + \frac{b_i}{i}z)^{k_i}$, we have $p_n(\lambda z) = \sum_{j=0}^{n-1}\lambda^{n-j}U_j$. For $j = 0, 1, 2, 3$, subsets $\sigma(n, n-j)$ of $\sigma(n)$ can be found readily. Table 1 displays the values of $k_1, k_2, \cdots, k_n$ for $j = 0, 1, 2, 3$.

Letting $\omega_i = \frac{a_i}{i\lambda}, \beta_i = \frac{b_i}{i}$, we have

$$
\begin{aligned}
U_0 =& \frac{(\omega_1+\beta_1 z)^n}{n!},\\
U_1 =& \frac{1}{(n-2)!}(\omega_1+\beta_1 z)^{n-2}(\omega_2+\beta_2 z),\\
U_2 =& \frac{1}{(n-3)!}(\omega_1+\beta_1 z)^{n-3}(\omega_3+\beta_3 z)+\\
& \frac{1}{(n-4)!2!}(\omega_1+\beta_1 z)^{n-4}(\omega_2+\beta_2 z)^2,\\
U_3 =& \frac{1}{(n-4)!}(\omega_1+\beta_1 z)^{n-4}(\omega_4+\beta_4 z)+\\
& \frac{1}{(n-5)!}(\omega_1+\beta_1 z)^{n-5}(\omega_2+\beta_2 z)(\omega_3+\beta_3 z)+\\
& \frac{1}{(n-6)!3!}(\omega_1+\beta_1 z)^{n-6}(\omega_2+\beta_2 z)^3.
\end{aligned}
$$

$k=n-j$	k_1	k_2	k_3	k_4	k_5	$\cdots$	k_n
$n-0$	n	0	0	0	0	$\cdots$	0
$n-1$	$n-2$	1	0	0	0	$\cdots$	0
$n-2$	$n-3$	0	1	0	0	$\cdots$	0
$n-2$	$n-4$	2	0	0	0	$\cdots$	0
$n-3$	$n-4$	0	0	1	0	$\cdots$	0
$n-3$	$n-5$	1	1	0	0	$\cdots$	0
$n-3$	$n-6$	3	0	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1

Now, $p_n(\lambda z)$ can be written

$$
p_n(\lambda z) = \frac{\lambda^n(\omega_1+\beta_1 z)^n}{n!} + \sum_{j=1}^{n-1}\lambda^{n-j}U_j, \tag{2.3}
$$

For $j \geq 1$, define $W_j = \frac{U_j}{U_0} = \frac{n!}{(\omega_1+\beta_1 z)^n}U_j$, with $W_0 = 1$. In particular

$$
\begin{aligned}
W_1 =&(\omega+\beta_1 z)^{-2}(\omega_2+\beta_2 z)(n)_2\\
W_2 =&(\omega+\beta_1 z)^{-3}(\omega_3+\beta_3 z)(n)_3 + \frac{1}{2}(\omega_1+\beta_2 z)^{-4}(\omega_2+\beta_2 z)^2(n)_4\\
W_3 =&(\omega+\beta_1 z)^{-4}(\omega_4+\beta_4 z)(n)_4 + (\omega_1+\beta_1 z)^{-5}(\omega_2+\beta_2 z)(\omega_3+\beta_3 z)(n)_5+\\
& \frac{1}{3!}(\omega_1+\beta_1 z)^{-6}(\omega_2+\beta_2 z)^3(n)_6.
\end{aligned}
$$

In general, for $j \geq 1$,

$$
\begin{aligned}
W_j =&(\omega_1+\beta_1 z)^{-(j+1)}(\omega_{j+1}+\beta_{j+1} z)(n)_{j+1} + \cdots +\\
& \frac{1}{j!}(\omega_1+\beta_2 z)^{-2j}(\omega_2+\beta_2 z)^j(n)_{2j},
\end{aligned} \tag{2.4}
$$

where $(n)_k := n(n-1)(n-2)\cdots(n-k+1), k \ge 1$, the kth falling factorial of n. We see that W_j is a polynomial in n of degree $2j$ with $W_0 = 1$. We may rewrite (2.3) as follows

$$p_n(\lambda z) = \frac{\lambda^n(\omega_1 + \beta_1 z)^n}{n!}[1 + \sum_{j=1}^{n-1} W_j(\frac{1}{\lambda})^j]. \tag{2.5}$$

Theorem 2.1 *Suppose that $p_n(z)$ has the C_n-representation*

$$p_n(z) = \frac{1}{n!}C_n(f_1, f_2, \cdots, f_n),$$

where the f_i's are linear functions of $z, i = 1, 2, \cdots, n$. Let n and λ become large such that $n = o(\lambda^{1/2})$ as $\lambda \to \infty$. Then an asymptotic expansion of $p_n(\lambda z)$ for any $z \in C$ with $|\omega_1 + \beta_1 z| > 0$ is given by

$$\frac{n!\lambda^{-n}}{(\omega_1 + \beta_1 z)^n} p_n(\lambda z) = 1 + \sum_{j=1}^{n-1} W_j \lambda^{-j}, \tag{2.6}$$

where W_j is given in (2.4).

Proof Let $w_m = \sum_{m+1}^{n-1} W_j \lambda^{-j}$. To prove the theorem, first, we have to show that $\{W_j(1/\lambda)^j\}$ is an asymptotic sequence as $\lambda \to \infty$, under the condition $n = o(\lambda^{1/2})$. But this follows easily since

$$\frac{W_{j+1}(\frac{1}{\lambda})^{j+1}}{W_j(\frac{1}{\lambda})^j} = \frac{1}{\lambda}\overline{O}(n^{2j+2})\overline{O}(n^{2j}) = \frac{1}{\lambda}\overline{O}(n^2) \to 0 \text{ as } \lambda \to \infty. \tag{2.7}$$

Second, we will show that for any $m, m \ge 1, w_m = o(W_m \lambda^{-m})$ as $\lambda \to \infty$ with $n = o(\lambda^{1/2})$. We do this as follows:

$$\begin{aligned}\frac{w_m}{W_m\lambda^{-m}} &= \frac{1}{\lambda}\frac{W_{m+1}}{W_m} + \frac{1}{\lambda^2}\frac{W_{m+2}}{W_m}\frac{1}{\lambda^3}\frac{W_{m+3}}{W_m} + \cdots \\ &= \frac{1}{\lambda}\frac{W_{m+1}}{W_m} + \frac{1}{\lambda^2}\frac{W_{m+1}}{W_m}\frac{W_{m+2}}{W_{m+1}} + \\ &\quad \frac{1}{\lambda^3}\frac{W_{m+1}}{W_m}\frac{W_{m+2}}{W_{m+1}}\frac{W_{m+3}}{W_{m+2}} + \cdots\end{aligned}$$

From (2.7) we have $\frac{1}{\lambda}\frac{W_{m+1}}{W_m} = O(\frac{n^2}{\lambda}), \frac{1}{\lambda^2}\frac{W_{m+2}}{W_{m+1}} = O(\frac{n^4}{\lambda^2}), \cdots$. Thus there exist constants $c_1, c_2, \cdots$ (constant with respect to λ, and n), such that

$$\begin{aligned}|\frac{w_m}{W_m(\frac{1}{\lambda})^m}| &= c_1\frac{n^2}{\lambda} + c_2(\frac{n^2}{\lambda})^2 + c_3(\frac{n^2}{\lambda})^3 + c_4(\frac{n^2}{\lambda})^4 + \cdots \\ &= \frac{n^2}{\lambda}[c_1 + c_2(\frac{n^2}{\lambda}) + c_3(\frac{n^2}{\lambda})^2 + c_4(\frac{n^2}{\lambda})^3 + \cdots].\end{aligned}$$

For any fixed $z \in C, |c_i| < K$, for some constant K, for each $i, i = 1, 2, \cdots$. Since $n = o(\lambda^{1/2})$ as $\lambda \to \infty$, it follows that $\frac{w_m}{W_m\lambda^{-m}} \to 0$ as $\lambda \to \infty$. □

3. Some lemmas

Throughout the paper we will make use of formal power series over the complex number field C. For any given power series $f(t)$ with $f(0) > 0$ we denote $\log f(t)$ by $\hat{f}(t)$, where the logarithm is taken to be the principal branch defined on $C - \{t : t \leq 0\}$. Also we write $f(k)$ in the power series form

$$f(t) = \sum_{n=0}^{\infty} \begin{bmatrix} f \\ n \end{bmatrix} t^n,$$

where $\begin{bmatrix} f \\ n \end{bmatrix}$ denotes the coefficient of t^n in the Maclaurin series expansion of f. These notations were adopted from [6].

The C_n-representation of a polynomial may be obtained using the following lemma (cf. Theorem 1 of [6]).

Lemma 3.1 *Let $\varphi(t)$ be the formal power series $\varphi(t) = \sum_{n=0}^{\infty} \begin{bmatrix} \varphi \\ n \end{bmatrix} t^n$ with $\varphi(0) > 0$. Suppose that $\hat{\varphi}(t)$ has the series expansion $\hat{\varphi}(t) = \sum_{n=0}^{\infty} \begin{bmatrix} \hat{\varphi}(t) \\ n \end{bmatrix} t^n$. Then*

$$\begin{bmatrix} \varphi \\ n \end{bmatrix} = \frac{\varphi(0)}{n!} C_n \left(1 \cdot \begin{bmatrix} \hat{\varphi} \\ 1 \end{bmatrix}, 2 \cdot \begin{bmatrix} \hat{\varphi} \\ 2 \end{bmatrix}, \cdots, n \cdot \begin{bmatrix} \hat{\varphi} \\ n \end{bmatrix} \right), \tag{3.1}$$

where $\begin{bmatrix} \hat{\varphi} \\ j \end{bmatrix}$ denotes the coefficient of t^j in the power series expansion of $\hat{\varphi}$.

The Poisson-Charlier, weighted Touchard, Toscano and Angelescu polynomials may be defined through their generating functions given, respectively, by equations (3.2)–(3.5) below.

$$e^t e^{z \log(1+t)} = \sum_{n=0}^{\infty} (PC)_n(z) t^n, \tag{3.2}$$

$$(1-t)^{-\rho} \exp[z(e^t - 1)] = \sum_{n=0}^{\infty} T_n^{\rho}(z) t^n, \quad \rho > 0, \tag{3.3}$$

$$e^{\rho t} \exp[z(1 - e^t)] = \sum_{n=0}^{\infty} (Tos)_n^{\rho}(z) t^n, \quad \rho > 0, \tag{3.4}$$

$$\frac{1}{(1+t)} \exp \frac{zt}{t-1} = \sum_{n=0}^{\infty} A_n(z) t^n. \tag{3.5}$$

Lemma 3.2 *Let $\rho > 0$ and $z \in C$. Then we have the C_n-representations*

$$(PC)_n(z) = \frac{1}{n!}C_n(1+z, -z, \cdots, (-1)^{n-1}z),$$
$$T_n^\rho(z) = \frac{1}{n!}C_n(\rho + \frac{z}{0!}, \rho + \frac{z}{1!}, \cdots, \rho + \frac{z}{(n-1)!}),$$
$$(Tos)_n^\rho(z) = \frac{1}{n!}C_n(\frac{\rho - z}{0!}, \frac{-z}{1!}, \cdots, \frac{-z}{(n-1)!}),$$
$$A_n(z) = \frac{1}{n!}C_n(-1-z, 1-2z, \cdots, (-1)^n - nz).$$

Proof We will derive the C_n-representation of the Poisson-Charlier polynomials $(PC)_n(z)$. The others can be done similarly. Let $\varphi(t) = e^t e^{z\log(1+t)}$. Then $\hat{\varphi}(t) = t + z\log(1+t)$. For $|t| < 1$, $\log(1+t) = t - \frac{t^2}{3} + \frac{t^3}{3} - \cdots$, hence $\hat{\varphi}(t) = t + z(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots)$. Thus

$$\left[\begin{array}{c}\hat{\varphi} \\ n\end{array}\right] = \begin{cases} 1+z, & \text{if } n = 1, \\ \frac{(-1)^{n-1}z}{n}, & \text{if } n > 1.\end{cases}$$

Applying Lemma 3.1, we have

$$(PC)_n(z) = \frac{1}{n!}C_n(1+z, -z, \cdots, (-1)^{n-1}z). \quad \square$$

The next two lemmas are useful in the estimation of remainders.

Lemma 3.3 *Let $1^{k_1}2^{k_2}\cdots n^{k_n} \in \sigma(n, n-j), 0 \le j \le n-1$. Then we have*

$$\sum_{\sigma(n,n-j)} \frac{n!}{k_1!k_2!\cdots k_n!} = \binom{n-1}{n-1-j}\frac{n!}{(n-j)!} < \frac{n^{2j}}{j!}.$$

Proof The equality in the lemma is an identity which may be found in [2] (cf. theroem B of §3.3). The inequality easily follows. □

Lemma 3.4 *Let $j \ge 1$ and let $1^{k_1}2^{k_2}\cdots n^{k_n} \in \sigma(n, n-j)$. Then $n - 2j \le k_1 \le n - j - 1$.*

4. The Asymptotic Expansions

Suppose $\rho > 0$ and $\rho = O(\lambda)$ as $\lambda \to \infty$. Let $\zeta = 1/\lambda, \nu = \rho/\lambda$. Now we will apply the discussion in section 2 to the polynomials with C_n-representation given in Lemma 3.2. We state the results in the following theorems.

Theorem 4.1 *Let n and λ become large such that $n = o(\lambda^{1/2})$ as $\lambda \to \infty$. Then for any given $m \ge 1$ and any $z \in C$ such that $|1 + \lambda z| > 0$ for large λ, an asymptotic expansion for the Poisson-Charlier polynomials is given by*

$$\frac{n!}{(1+\lambda z)^n}(PC)_n(\lambda z) = 1 + \sum_{j=1}^{m} B_j \lambda^{-j} + b_m, \tag{4.1}$$

where

$$B_j = \frac{n!}{(\zeta + z)^n} \sum_{\sigma(n,n-j)} \frac{(\zeta + z)^{k_1}(-z/2)^{k_2} \cdots ((-1)^{n-1} z/n)^{k_n}}{k_1!k_2! \cdots k_n!}, \tag{4.2}$$

and

$$b_m = \sum_{j=m+1}^{n-1} B_j \lambda^{-j} = o(B_m \lambda^{-m}) \tag{4.3}$$

as $\lambda \to \infty$.

Proof A Poisson-Charlier polynomial of degree n has the C_n-representation

$$\begin{aligned}(PC)_n(\lambda z) &= \frac{1}{n!} C_n(1 + \lambda z, -\lambda z, \cdots, (-1)^{n-1}\lambda z) \\ &= \frac{\lambda^n(\zeta + z)^n}{n!} + \sum_{j=1}^{n-1} \lambda^{n-j} \sum_{\sigma(n,n-j)} \frac{(\zeta + z)^{k_1}(-z/2)^{k_2} \cdots ((-1)^{n-1} z/n)^{k_n}}{k_1!k_2! \cdots k_n!}.\end{aligned}$$

Taking $U_j = \sum_{\sigma(n,n-j)} \frac{(\zeta+z)^{k_1}(-z/2)^{k_2}\cdots((-1)^{n-1}z/n)^{k_n}}{k_1!k_2!\cdots k_n!}$, $j \geq 0$, we have $W_j = \frac{n!}{(\zeta+z)^n} U_j$. By (2.5) we may write $(PC)_n(\lambda z) = \frac{\lambda^n(\zeta+z)^n}{n!}[1+\sum_{j=1}^m B_j\lambda^{-j}+b_m]$, where $B_j = \frac{n!}{(\zeta+z)^n} \sum_{\sigma(n,n-j)} \frac{(\zeta+z)^{k_1}(-z/2)^{k_2}\cdots((-1)^{n-1}z/n)^{k_n}}{k_1!k_2!\cdots k_n!}$. Now the theorem follows from Theorem 2.1. □

The next theorems are proved similarly.

Theorem 4.2 *Let n and λ become large such that $n = o(\lambda^{1/2})$ as $\lambda \to \infty$. Then for any given $m \geq 1$ and any $z \in C$ such that $|\rho + \lambda z| > 0$ for large λ, an asymptotic expansion for the weighted Touchard polynomials is given by*

$$\frac{n!}{(\rho + \lambda z)^n} T_n^{(\rho)}(\lambda z) = 1 + \sum_{j=1}^{m} D_j \lambda^{-j} + d_m, \tag{4.4}$$

where

$$D_j = \frac{n!}{(\nu + z)^n} \sum_{\sigma(n,n-j)} \frac{(\nu + z)^{k_1}(\nu/2 + z/2)^{k_2} \cdots (\nu/n + z/n!)^{k_n}}{k_1!k_2! \cdots k_n!}, \tag{4.5}$$

and

$$d_m = \sum_{j=m+1}^{n-1} D_j \lambda^{-j} = o(D_m \lambda^{-m}) \tag{4.6}$$

as $\lambda \to \infty$.

Theorem 4.3 *Let n and λ become large such that $n = O(\lambda^{1/2})$ as $\lambda \to \infty$. Then for any given $m \geq 1$ and any $z \in C$ such that $|\rho + \lambda z| > 0$ for large λ, an asymptotic expansion formula for the Toscano polynomials is given by*

$$\frac{n!}{(\rho - \lambda z)^n} (Tos)_n^{\rho}(\lambda z) = 1 + \sum_{j=1}^{m} E_j \lambda^{-j} + e_m, \tag{4.7}$$

where

$$E_j = \frac{n!}{(\nu - z)^n} \sum_{\sigma(n,n-j)} \frac{(\nu - z)^{k_1}(-z/2)^{k_2}\cdots(-z/n!)^{k_n}}{k_1!k_2!\cdots k_n!}, \tag{4.8}$$

and

$$e_m = \sum_{j=m+1}^{n-1} E_j\lambda^{-j} = o(E_m\lambda^{-m}) \tag{4.9}$$

as $\lambda \to \infty$.

Theorem 4.4 *Let n and λ become large such that $n = o(\lambda^{1/2})$ as $\lambda \to \infty$. Then for any given $m \geq 1$ and any $z \in C$ such that $|1 + \lambda z| > 0$ for large λ, an asymptotic expansion formula for the Angelescu polynomials is given by*

$$\frac{n!}{(-1 - \lambda z)^n} A_n(\lambda z) = 1 + \sum_{j=1}^{m} F_j\lambda^{-j} + f_m, \tag{4.10}$$

where

$$F_j = \frac{n!}{(-\zeta - z)^n} \sum_{\sigma(n,n-j)} \frac{(-\zeta - z)^{k_1}(\zeta/2 - z)^{k_2}\cdots((-1)^n\zeta/n - z)^{k_n}}{k_1!k_2!\cdots k_n!}, \tag{4.11}$$

and

$$f_m = \sum_{j=m+1}^{n-1} F_j\lambda_{-j} = o(F_m\lambda^{-m}) \tag{4.12}$$

as $\lambda \to \infty$.

5. Remainder estimates

The next theorems give estimates of the remainders defined in Theorems 4.1–4.4, within some specified domains.

Theorem 5.1 *Let n and λ become large such that $n = o(\lambda^{1/2})$ as $\lambda \to \infty$. Then for any given $m \geq 1$ and for sufficiently large λ, the remainder b_m defined by (4.3) satisfies*

$$|b_m| < \frac{3}{2}\frac{|\zeta + z|^{-(m+1)}}{(m+1)!}(\frac{n^2}{\lambda})^{m+1}, \tag{5.1}$$

for any nonzero z with Re$z > -\zeta/2$, *and*

$$|b_m| < \frac{3}{2}\frac{|z|/|\zeta + z|^2)^{m+1}}{(m+1)!}(\frac{n^2}{\lambda})^{m+1}, \tag{5.2}$$

for any $z \neq -\zeta$ with Re$z < -\zeta/2$.

Proof Notice that for nonzero z with Re$z > -\zeta/2$, $|\zeta + z| > |z|$, Thus,

$$|B_j| \leq \frac{n!}{|\zeta + z|^n} \sum_{\sigma(n,n-j)} \frac{|\zeta + z|^{k_1+k_2+\cdots+k_n}}{k_1!k_2!\cdots k_n!} = \frac{|\zeta + z|^{n_j}}{|\zeta + z|^n} \sum_{\sigma(n,n-j)} \frac{n!}{k_1!k_2!\cdots k_n!}.$$

Using Lemma 3.3, $|B_j| < \frac{1}{|\zeta+z|^j}\frac{n^{2j}}{j!}$. Consequently,

$$|b_m| \le \sum_{j=m+1}^{n-1} |B_j|\lambda^{-j} < \sum_{j=m+1}^{n-1} |\frac{|\zeta+z|^{-j}}{j!}(\frac{n^2}{\lambda})^j$$
$$< \frac{|\zeta+z|^{-(m+1)}}{(m+1)!}(\frac{n^2}{\lambda})^{m+1}[1+\frac{\mu}{m+2}+\frac{\mu^2}{(m+3)(m+2)}+\cdots],$$

where $\mu = |\zeta+z|^{-1}(\frac{n^2}{\lambda})$. Since $n = o(\lambda^{1/2})$ as $\lambda \to \infty, \mu < 1$ for sufficiently large λ with z fixed. Hence,

$$|b_m| < \frac{3}{2}\frac{|\zeta+z|^{-(m+1)}}{(m+1)!}(\frac{n^2}{\lambda})^{m+1}.$$

Next we consider the case when $\mathrm{Re}z < -\zeta/2$. Notice that $|z| > |\zeta+z|$, whenever $\mathrm{Re}z < -\zeta/2$. Thus, for $z \ne -\zeta, \mathrm{Re}z < \frac{-\zeta}{2}$.

$$|B_j| \le \frac{n!}{|\zeta+z|^n}\sum_{\sigma(n,n-j)}\frac{|\zeta+z|^{k_1}|z|^{k_2+k_3+\cdots+k_n}}{k_1!k_2!\cdots k_n!}.$$

Let $J = |\zeta+z|^{k_1}|z|^{k_2+k_3+\cdots+k_n}$. By Lemma 3.4, $k_1 = n-2j+d$ for some integer $d \ge 0$, hence

$$J = |\zeta+z|^{n-2j+d}|z|^{n-j-(n-2j+d)} = |\zeta+z|^{n-2j+d}|z|^{j-d}$$
$$= |\zeta+z|^{n-2j}|z|^j\frac{|\zeta+z|^d}{|z|^d} \le |\zeta+z|^{n-2j}|z|^j.$$

This gives $|B_j| \le (\frac{|z|}{|\zeta+z|^2})^j\sum_{\sigma(n,n-j)}\frac{n!}{k_1!k_2!\cdots k_n!} < (\frac{|z|}{|\zeta+z|^2})^j\frac{n^{2j}}{j!}$. Consequently,

$$|b_m| \le \frac{n-1}{j=m+1}|B_j|(\frac{1}{\lambda})^j < \sum_{j=m+1}^{n-1}(\frac{|z|}{|\zeta+z|^2})^j\frac{1}{j!}(\frac{n^2}{\lambda})^j$$
$$= (\frac{|z|}{|\zeta+z|^2})^{m+1}\frac{1}{(m+1)!}(\frac{n^2}{\lambda})^{m+1}[1+\frac{\alpha}{m+2}+\frac{\alpha^2}{(m+3)(m+2)}+\cdots],$$

where $\alpha = \frac{|z|}{|\zeta+z|^2}(\frac{n^2}{\lambda})$. Since $\alpha < 1$ for sufficiently large λ,

$$|b_m| < \frac{3}{2}(\frac{|z|}{|\zeta+z|^2})^{m+1}\frac{1}{(m+1)!}(\frac{n^2}{\lambda})^{m+1},$$

which is the desired result. □

Theorem 5.2 *Let n and λ become large such that $n = o(\lambda^{1/2})$ as $\lambda \to \infty$. Then for any given $m \ge 1$ and for sufficiently large λ, the remainder d_m defined by (4.6) satisfies*

$$|d_m| < \frac{3}{2}\frac{|\nu+z|^{-(m+1)}}{(m+1)!}(\frac{n^2}{\lambda})^{m+1}, \quad (5.3)$$

for all z such that $\mathrm{Re}z > 0$. *If* $\mathrm{Re}z < 0, z \neq \nu$, *(5.3) still holds for* $\lambda > \rho/|z|$.

Proof If $\mathrm{Re}z > 0$,

$$|\nu + z| > |\nu + \frac{z}{k!}| > |\frac{\nu}{k} + \frac{z}{k!}| \tag{5.4}$$

for all $k, k = 2, 3, \cdots$. Thus,

$$|D_j| \leq \frac{n!}{|\nu + z|^n} \sum_{\sigma(n,n-j)} \frac{|\nu + z|^{k_1+k_2+\cdots+k_n}}{k_1!k_2!\cdots k_n!} = \frac{|\nu + z|^{n-j}}{|\nu + z|^n} \sum_{\sigma(n,n-j)} \frac{n!}{k_1!k_2!\cdots k_n!}.$$

Using Lemma 3.3, $|D_j| < \frac{1}{|\nu+z|^j}\frac{n^{2j}}{j!}$. Consequently,

$$\begin{aligned} |d_m| &\leq \sum_{j=m+1}^{n-1} |D_j|\lambda^{-j} < \sum_{j=m+1}^{n-1} \frac{|\nu + z|^{-j}}{j!}(\frac{n^2}{\lambda})^j \\ &< \frac{|\nu + z|^{-(m+1)}}{(m+1)!}(\frac{n^2}{\lambda})^{m+1}[1 + \frac{\beta}{m+2} + \frac{\beta^2}{(m+3)(m+2)} + \cdots], \end{aligned}$$

where $\beta = |\nu + z|^{-1}(\frac{n^2}{\lambda})$. Since $n = o(\lambda^{1/2})$ as $\lambda \to \infty, \beta < 1$ for sufficiently large λ, with z fixed. Hence

$$|d_m| < \frac{3}{2}\frac{|\nu + z|^{-(m+1)}}{(m+1)!}(\frac{n^2}{\lambda})^{m+1}.$$

If $\mathrm{Re}z < 0, z \neq -\nu$, we only need to show that the first inequality in (5.4) is true when $\lambda > \rho/|z|$. To do this let $z = re^{i\theta}$ and solve for λ the inequality

$$|\nu + re^{i\theta}| > |\nu + re^{i\theta}/k!|. \quad \square$$

Theorem 5.3 *Let n and λ become large such that $n = o(\lambda^{1/2})$ as $\lambda \to \infty$. Then for any given $m \geq 1$ and for sufficiently large λ, the remainder e_m defined by (4.9) satisfies*

$$|e_m| < \frac{3}{2}\frac{|\nu - z|^{-(m+1)}}{(m+1)!}(\frac{n^2}{\lambda})^{m+1}, \tag{5.5}$$

for all nonzero z such that $\mathrm{Re}z < \nu/2$, *and*

$$|e_m| < \frac{3}{2}\frac{|z|/|\nu - z|)^{m+1}}{(m+1)!}(\frac{n^2}{\lambda})^{m+1}, \tag{5.6}$$

for all $z \neq \nu$ with $\mathrm{Re}z > \nu/2$.

Proof It is clear that for all nonzero z with $\Re z < \nu/2$ we have

$$|\nu - z| > |z| > |z/k!|, \quad \forall k = 2, 3, 4, \cdots$$

and when $\mathrm{Re}z > \nu/2$ with $z \neq \nu, |z| > |\nu - z|$. Now the proof follows similar arguments as that of Theorem 5.1. $\square$

Theorem 5.4 *Let n become large such that $n = o(\lambda^{1/2})$ as $\lambda \to \infty$. Then for any given $m \geq 1$ and for sufficiently large λ, the remainder f_m defined by (4.12) satisfies*

$$|f_m| < \frac{3}{2}\frac{|z+\zeta|^{-(m+1)}}{(m+1)!}(\frac{n^2}{\lambda})^{m+1}, \tag{5.7}$$

for all z with $\mathrm{Re} z > 0$, *and*

$$|f_m| < \frac{3}{2}\frac{|z-\zeta/2|/|z+\zeta|^2)^{m+1}}{(m+1)!}(\frac{n^2}{\lambda})^{m+1}, \tag{5.8}$$

for all z with $\mathrm{Re} z < \zeta$.

Proof It can be seen easily that for all z with $\Re z > 0$, $|z+\zeta| > |z \pm \zeta/k|$, $\forall k = 2, 3, \cdots$, and for all z with $\mathrm{Re} z < -\zeta$, we have

$$|z - \zeta/2| > |z+\zeta|, \quad |z-\zeta/2| \geq |z \pm \zeta/k|, \quad \forall k = 2, 3, \cdots.$$

The proof follows similarly as in Theorem 5.1. □

Remarks Some applicability and limitation of the asymptotic formulas (4.1), (4.4), (4.7) and (4.10) may be illustrated by mentioning a few examples. Taking $\lambda = n\Gamma(n+1)$ with $\Gamma(z)$, the gamma function, and let $z \in C$ with $|z| = 1$,

$$\zeta = \frac{1}{\lambda} \to 0, \frac{n^2}{\lambda} = \frac{n^2}{n\Gamma(n+1)} = \frac{1}{(n-1)!} \to 0, \quad \text{as} \quad n \to \infty$$

and $|\zeta + z| \sim 1$ as $n \to \infty$, so that the asymptotic expansion, for $(PC)_n(\lambda z)$ and $A_n(\lambda z)$ may be obtained via (4.1) and (4.10), respectively, with remainder estimates of order $O((\frac{n^2}{n\Gamma(n+1)})^{m+1}) = O(((n-1)!)^{-(m+1)})$.

Similarly, taking $\rho = n, \lambda = n^2 \log n, z \in C$ with $|z| = 1$, we have

$$\nu = \frac{n}{n^2 \log n} = \frac{1}{n \log n} \sim 0 \quad \text{as} \quad n \to \infty,$$

$$\frac{n^2}{\lambda} = \frac{1}{\log n} \to 0 \quad \text{as} \quad n \to \infty,$$

and $|\nu + z| \sim 1, |\nu - z| \sim 1$ as $n \to \infty$, so that the asymptotic expansion for $T_n^{(n)}(zn^2 \log n)$ and $Tos_n^{(n)}(zn^2 \log n)$ can be found via (4.4) and (4.7), respectively, with remainder estimates of order $O((\log n)^{-(m+1)})$.

If $\lambda = |z|$ the asymptotic expansions hold true whenever $n = o(|z|^{1/2})$ as $|z| \to \infty$ along a fixed direction θ.

However, the asymptotic formulas obtained in this paper do not apply when $n = \lambda$.

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Sheffer 多项式的渐近展开

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摘 要: 本文利用组合分析中的循环指示表示方法, 找到了 Sheffer 型多项式的渐近展开公式及余项估计. 文末讨论了所得渐近公式的运用范围.