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Zeros of Perturbed m-Accretive Operators *

ZHANG Guo-wei[1], *HUI Shu-rong*[2]

(1. Dept. of Math., Northeastern University, Shenyang 110006, China;
2. Dept. of Basi. Sci., Shenyang Agricultural University, Shenyang 110161, China)

Abstract: In this paper, we give a full answer to a question posed by Kartsatos[1], and weaken the conditions of Kartsatos' theorem.

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1. Introduction

Let X stand for a real Banach space with norm $\|\cdot\|$ and normalized duality J. The symbols ∂D and $\overline{D}$ denote the boundary and closure of the set D, respectively. An operator $T: D(T)\subset X\to 2^X$ is accretive if for every $x,y\in D(T)$ there exists $j\in J(x-y)$ such that $(u-v,j)\geq 0$ for every $u\in Tx, v\in Ty$. An accretive operator T is called m-accretive if $R(T+\lambda I)=X$ for every $\lambda>0$, where I denotes the identity operator on X. It is called m-dissipative if $-T$ is m-accretive. For an m-accretive operator T, the resolvents $J_\lambda: X\to D(T)$ of T are defined by $J_\lambda=(I+\lambda T)^{-1}$ for all $\lambda\in(0,\infty)$. J_λ is a nonexpansive mapping on X for all $\lambda>0$. The operator $T_\lambda=\frac{1}{\lambda}(I-J_\lambda)$is a globally Lipschitz mapping with $T_\lambda x\in TJ_\lambda x$ for every $x\in X$. An operator $C: D(C)\subset X\to X$ is compact if it is continuous and maps bounded subsets of $D(C)$ onto relatively compact sets. We denote by $B_r(0)$ the open ball of X with center at zero and radium $r>0$.

Kartsatos noted in [1] that it would be very interesting to know anything about the relationship between the boundary condition

$$[I-(T+C)](D(T)\cap\partial B_r(0))\subset\overline{B_r(0)}, \tag{1}$$

and the following inner product condition: for every $x\in D(T)\cap\partial B_r(0)$ there exists $j\in Jx$ such that

$$(u+Cx,j)\geq 0 \tag{2}$$

*Received date: 1998-12-03
Biography: ZHANG Guo-wei (1965-), male, M.Sc., associate professor.
E-mail: zhanggw@sy163.net

for every $u \in Tx$. This last boundary condition and its other extensions provide the standard criteria for the existence of zeros of the operator $T+C$.

In this paper, we first point out that (1) actually is a norm condition and (2) can be deduced from (1), but the converse is not necessarily true. In this sense, the norm condition is stronger than the inner condition. After that, we give under an inner condition an existence theorem of zeros without the convex set condition in Theorem 1 in [1].

2. Main results

Theorem 1 *Let $T: D(T) \subset X \to 2^X, C: \overline{B_r(0)} \to X$. If*

$$[I-(T+C)](D(T)\cap \partial B_r(0)) \subset \overline{B_r(0)} \tag{1}$$

then for every $x \in D(T)\cap \partial B_r(0)$ there exists $j \in Jx$ such that

$$(u+Cx, j) \geq 0 \tag{2}$$

for every $u \in Tx$. Conversely, it is not necessarily true.

Proof For every $x \in D(T)\cap\partial B_r(0)$ and every $u \in Tx$, we have from (1) that $\|x-u-Cx\| \leq r$, this means that (1) is a norm condition.

If there exist $x_0 \in D(T)\cap \partial B_r(0)$ and $u_0 \in Tx_0$ such that $(u_0+Cx_0, j) < 0$ for every $j \in Jx_0$, then $\|x_0-u_0-Cx_0\|\,\|x_0\| \geq (x_0-u_0-Cx_0, j) > (x_0, j) = \|x_0\|^2 = r^2$, and hence $\|x_0-u_0-Cx_0\| > r$, which is a contradiction.

Conversely, consider a counterexample. Let $T: (-\frac{\pi}{2}, \frac{\pi}{2}) \to R$ be defined by $Tx = tgx$ which is m-accretive and $C=0$ which is compact. It is obvious that there exists $x_0 \in (\frac{\pi}{4}, \frac{\pi}{2})$ such that $|x_0 - tgx_0| = tgx_0 - x_0 > x_0$, but $x_0 tgx_0 > 0$ and $(-x_0)tg(-x_0) > 0$. □

Theorem 2 *Let G be a bounded, open subset of X. Let $T: D(T) \subset X \to 2^X$ be m-accretive and $C: \overline{G} \to X$ compact. Assume that $x_0 \in D(T)\cap G$ and for every $x \in D(T)\cap \partial G$, there exists $j \in J(x-x_0)$ such that*

$$(u+Cx, j) \geq 0 \tag{3}$$

for every $u \in Tx$, then $0 \in \overline{R(T+C)}$. If instead of the compactness of C, let $J_1 = (I+T)^{-1}$ be compact and C be continuous and bounded, then $0 \in R(T+C)$.

Proof We may assume that $x_0 = 0$ and $0 \in T0$. In fact, if one of these is not true, we may consider instead the operators $\tilde{T}$and $\tilde{C}$ defined on $D(\tilde{T})$ and $\tilde{\overline{G}}$, respectively, where $D(\tilde{T}) = D(T)-x_0, \tilde{G} = G-x_0, \tilde{T}x = T(x+x_0)-v_0, \tilde{C}x = C(x+x_0)+v_0$. Here v_0 is a fixed point in Tx_0. It can be inferred from the proof of Theorem 2 in [2] that $\tilde{J}_1 = (I+\tilde{T})^{-1}$ is compact when J_1 is compact.

Consider the homotopy mapping $H(t,x) = (tnT+I)^{-1}(-tnCx), (t,x) \in [0,1]\times\overline{G}$, we shall show that (i) $H(t,\cdot): \overline{G} \to X$ is compact for each $t \in [0,1]$; (ii) $H(t,x)$ is continuous, w. r. t. t at $t_0 \in [0,1]$, uniformly w. r. t. $x \in \overline{G}$; (iii) $H: [0,1]\times\overline{G} \to X$ is continuous. To do this, we claim that

$$\|(tnT+I)^{-1}(-tnCx) - (t_0nT+I)^{-1}(-tnCx)\| \leq \frac{2|t-t_0|}{t_0}\|tnCx\| \text{ for } t_0 \in (0,1].$$

In fact, let $y_{t_0} = (t_0 nT + I)^{-1}(-tnCx), y = (tnT + I)^{-1}(-tnCx)$, thus

$$\begin{aligned} y_{t_0} &= -tnCx - t_0 nT_{t_0 n}(-tnCx), \\ y_t &= -tnCx - tnT_{tn}(-tnCx), \\ y_t - y_{t_0} &= -n(tT_{tn}(-tnCx) - t_0 T_{t_0 n}(-tnCx). \end{aligned}$$

Since $T_{tn}(-tnCx) \in TJ_{tn}(-tnCx) = Ty_t, T_{t_0 n}(-tnCx) \in TJ_{t_0 n}(-tnCx) = Ty_{t_0}$, and T is accretive, we can select $j \in J(y_t - y_{t_0})$ such that $(T_{tn}(-tnCx) - T_{t_0 n}(-tnCx), j) \geq 0$, then

$$\begin{aligned} \|y_t - y_{t_0}\|^2 &= -n(tT_{tn}(-tnCx) - t_0 T_{t_0 n}(-tnCx), j) \\ &= -nt(T_{tn}(-tnCx) - T_{t_0 n}(-tnCx), j) - n(tT_{t_0 n}(-tnCx) - t_0 T_{t_0 n}(-tnCx), j) \\ &\leq n|t - t_0| \|T_{t_0 n}(-tnCx)\| \|y_t - y_{t_0}\|. \end{aligned}$$

By $\|T_{t_0 n}(-tnCx)\| = \|T_{t_0 n}(-tnCx) - T_{t_0 n}(0)\| \leq \frac{2}{t_0 n}\|tnCx\|$, we have

$$\|y_t - y_{t_0}\| < \frac{2|t - t_0|}{t_0}\|tnCx\|.$$

Now, (i) follows immediately. And by the claim, for $t_0 \in (0, 1]$,

$$\begin{aligned} \|H(t, x) - H(t_0, x)\| &\leq \|(tnT + I)^{-1}(-tnCx) - (t_0 nT + I)^{-1}(-tnCx)\| + \\ &\quad \|(t_0 nT + I)^{-1}(-tnCx) - (t_0 nT + I)^{-1}(-t_0 nCx)\| \\ &\leq \frac{2|t - t_0|}{t_0}\|tnCx\| + |t - t_0| \|Cx\|. \end{aligned}$$

Since G is bounded, $t \in [0, 1]$, C is compact and $H(0, x) = 0$, it follows that $H(t, x)$ is continuous w. r. t. t at $t_0 \in [0, 1]$, uniformly w. r. t. $x \in \overline{G}$; (iii) It is obvious that $H(t, x)$ is continuous w. r. t. x, so at $(t_0, x_0) \in [0, 1] \times \overline{G}$, for every $\varepsilon > 0$, there exists $\delta_1(t_0, x_0) > 0$ such that $\|H(t_0, x) - H(t_0, x_0)\| < \frac{\varepsilon}{2}$ whenever $x \in \overline{G}$ and $\|x - x_0\| < \delta_1(t_0, x_0)$. We know from (ii) that there exists $\delta_2(t_0) > 0$ such that $\|H(t, x) - H(t_0, x)\| < \frac{\varepsilon}{2}$ whenever t$\in [0, 1]$ and $|t - t_0| < \delta_2(t_0)$. Let $\delta = \min\{\delta_1, \delta_2\}$, then when $|t - t_0| < \delta$, and $\|x - x_0\| < \delta$, we have $\|H(t, x) - H(t_0, x_0)\| < \varepsilon$.

Now we show that the equation $x - H(t, x) = 0$ has no solution $x_t \in \partial G$ for any $t \in [0, 1)$. Assume that $x_t = H(t, x_t)$ for some $t \in [0, 1)$ and some $x_t \in \partial G$. Since $t = 0$ is impossible and $x_t \in \partial G$ implies $x_t \neq 0$, we have $t \in (0, 1)$. Also $x_t \in D(T) \cap \partial G$ and for some $u_t \in Tx_t$, $tu_t + \frac{1}{n}x_t + tCx_t = 0$. From (3), there exists $j \in Jx_t$ such that $\frac{1}{n}\|x_t\|^2 \leq t(x_t + Cx_t, j) + \frac{1}{n}\|x_t\|^2 = 0$, which is a contradiction to the fact that $x_t \in \partial G$.

If $x - H(1, x) = 0$ has a solution $x \in \partial G$, $Tx + Cx + \frac{1}{n}x \ni 0$. Otherwise, $x - H(1, x) = 0$ has no solution $x \in \partial G$, thus from [3, P.144], we have that Leray-Schauder degree $d(I - H(1, \cdot), G, 0) = d(I - H(0, \cdot), G, 0) = 1$ and the equation $x - H(1, x) = 0$ is solvable in G. This implies that $Tx + Cx + \frac{1}{n}x \ni 0$ is solvable in $D(T) \cap \overline{G}$. We denote the solution by x_n and let $n \to \infty$ ($\{x_n\}$ is bounded), then $0 \in \overline{R(T + C)}$.

On the other hand, instead of the compactness of C, let $J_1 = (I + T)^{-1}$ be compact and $C : \overline{G} \to X$ continuous and bounded. We observe that the resolvent identity $J_\lambda x = J_1 \left(\frac{1}{\lambda} x + \frac{\lambda - 1}{\lambda} J_\lambda x \right)$ implies that J_λ is compact for $\lambda > 0$. Thus, the entire proof above goes through with no change. We complete the proof by lemma 2 in [1]. □

Corollary 3 *Let G be an open, bounded subset of X. Let $T : D(T) \subset X \to 2^X$ be m-dissipative with $(I - T)^{-1}$ compact, $C : \overline{G} \to X$ continuous and bounded. Assume that $0 \in D(T) \cap G$ and for every $x \in D(T) \cap \partial G$, there exists $j \in Jx$ such that $(u + Cx, j) \leq 0$ for every $u \in Tx$, then the operator $T + C$ has a fixed point in $D(T) \cap \overline{G}$, i. e., there exists $x \in D(T) \cap \overline{G}$ such that $(T + C)x \ni x$.*

Proof Theorem 2(with $I - T$ in place of T and $-C$ in place of C) implies that the operator $I - (T + C)$ has a zero in $D(T) \cap \overline{G}$. This completes the proof.

A real number λ is called an eigenvalue of a pair of operators (T, C) if the equation $\lambda Tx + Cx \ni 0$ is solvable in $D(T) \cap D(C)$.

Corollary 4 *Let G be a bounded open subset of X. Let $T : D(T) \subset X \to 2^X$ be m-accretive with $0 \in D(T) \cap G$ and $0 \in T0$, $J_1 = (I + T)^{-1}$ compact, $C : \overline{G} \to X$ continuous and bounded. Assume that for every $x \in D(T) \cap \partial G$, there exists $j \in Jx$ such that $(u + Cx, j) \geq 0$ for every $u \in Tx$, then $0 \in R((\lambda + 1)T + C)(\lambda > 0)$, i. e., the number $\lambda + 1$ is an eigenvalue for the pair (T, C).*

Proof The result follows from Theorem 2 (with $(\lambda + 1)T$ instead of T).

References:

[1] KARTSATOS A G. *New results in the perturbation theory of maximal monotone and m-accretive operators in Banach spaces* [J]. Trans. Amer. Math. Soc., 1996, **348**: 1663-1707.

[2] KARTSATOS A G. *On the perturbation theory of m-accretive operators in Banach spaces* [J]. Proc. Amer. Math. Soc., 1996, **124**: 1811-1820.

[3] GUO D. *Nonlinear Functional Analysis* [M]. Shandong Sci. Technol. Press, Jinan, 1985.

[4] BARBU V. *Nonlinear Semigroups and Differential Equations in Banach Spaces* [M]. Noordhoff Int. Publ., Leyden(The Netherlands), 1975.

扰动 m- 增生算子的零集

张 国 伟[1]， 惠 淑 荣[2]

(1. 东北大学数学系，沈阳 110006; 2. 沈阳农业大学基础部，沈阳 110161)

摘 要: 本文对 Kartsatos 提出的一个问题给出了完整的回答并减弱了 Kartsatos 定理的条件.