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Maximum $K_{2,3}$-Packing Designs and Minimum $K_{2,3}$-Covering Designs of λK_v

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Abstract: Let G be a finite simple graph. A G-design (G-packing design, G-covering design) of λK_v, denoted by (v, G, λ)-GD ((v, G, λ)-PD, (v, G, λ)-CD), is a pair $(X, \mathcal{B})$ where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called blocks, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly (at most, at least) λ blocks of $\mathcal{B}$. A packing (covering) design is said to be maximum (minimum) if no other such packing (covering) design has more (fewer) blocks. In this paper, we determine the existence spectrum for the $K_{2,3}$-designs of λK_v, $\lambda > 1$, and construct the maximum packing designs and the minimum covering designs of λK_v with $K_{2,3}$ for any integer λ.

Key words: G-design; G-packing design; G-covering design.
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1. Introduction

A complete multigraph of order v and index λ, denoted by λK_v, is a graph with v vertices, where any two distinct vertices x and y are joined by λ edges (x, y). Let G be a finite simple graph. *A G-design* (*G-packing design, G-covering design*) of λK_v, denoted by (v, G, λ)-GD ((v, G, λ)-PD, (v, G, λ)-CD), is a pair $(X, \mathcal{B})$ where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly (at most, at least) λ blocks of $\mathcal{B}$. A packing (covering) design is said to be *maximum* (*minimum*) if no other such packing (covering) design has more (fewer) blocks. The number of blocks in a maximum packing design (minimum covering design), denoted by $p(v, G, \lambda)$ $(c(v, G, \lambda))$, is called the *packing* (*covering*) *number*. It is well known that

$$p(v, G, \lambda) \leq \lfloor \frac{\lambda v(v-1)}{2e(G)} \rfloor \leq \lceil \frac{\lambda v(v-1)}{2e(G)} \rceil \leq c(v, G, \lambda),$$

where $e(G)$ denotes the number of edges in G, $\lfloor x \rfloor$ denotes the greatest integer y such that $y \leq x$ and $\lceil x \rceil$ denotes the least integer y such that $y \geq x$. A (v, G, λ)-PD ((v, G, λ)-CD) is called to be *optimal* and denoted by (v, G, λ)-OPD ((v, G, λ)-OCD) if the left (right) equality holds. Obviously, there exists a (v, G, λ)-GD if and only if $p(v, G, \lambda) = c(v, G, \lambda)$ and a (v, G, λ)-GD can be regarded as (v, G, λ)-OPD or (v, G, λ)-OCD.

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The leave-edge graph $L_\lambda(\mathcal{D})$ of a packing design $\mathcal{D}$ is a subgraph of λK_v and its edges are the supplement of $\mathcal{D}$ in λK_v. The number of edges in $L_\lambda(\mathcal{D})$ is denoted by $|L_\lambda(\mathcal{D})|$. Especially, when $\mathcal{D}$ is maximum, $|L_\lambda(\mathcal{D})|$ is called *leave-edge number* and is denoted by $l_\lambda(v)$. Similarly, the *repeat-edge graph* $R_\lambda(\mathcal{D})$ of a covering design $\mathcal{D}$ is a subgraph of λK_v and its edges are the supplement of λK_v in $\mathcal{D}$. When $\mathcal{D}$ is minimum, $|R_\lambda(\mathcal{D})|$ is called the *repeat-edge number* and is denoted by $r_\lambda(v)$. Generally, the symbols $L_\lambda(\mathcal{D})$, $l_\lambda(v)$, $R_\lambda(\mathcal{D})$ and $r_\lambda(v)$ can be denoted by L_λ, l_λ, R_λ and r_λ, briefly. It is not difficult to show the following proposition:

Proposition 1.1[10] *If there exists a* $(v, K_{2,3}, \lambda)$*-GD, then* $p(v, K_{2,3}, \lambda) = c(v, K_{2,3}, \lambda) = \frac{\lambda v(v-1)}{12}$, *i.e.,* $l_\lambda = r_\lambda = 0$. *Else,*

$$l_\lambda = \lambda v(v-1)/2 - 6p(v, K_{2,3}, \lambda) > 0 \ \text{ and } \ r_\lambda = 6c(v, K_{2,3}, \lambda) - \lambda v(v-1)/2 > 0.$$

The G-packing and G-covering problems have attracted much attention in the last fifty years. Numerous papers were written on these subjects. In the last few years, the G-packing problems with five vertices have been determined. What about the following graphs G are known[13]:

1. Forest of order five, by Y. Roditty, 1986.
2. $G = K_5$, by J. Yin, 1994.
3. Two triangles with a common vertex, by E. J. Billington and C. C. Lindner, 1998.
4. Stars of five vertices plus one edge, by G. Ge, 1999.
5. Graph of five vertices having pendant point and six edges or less, by S. Zhang, 1999.

After these known results, the remaining graphs with five vertices and six edges are only $K_{2,3}$ and C_5 with a chord. Here and below, P_n denotes a path with n vertices, C_n denotes a cycle with length n, K_n denotes a complete graph with n vertices and $K_{m,n}$ denotes a complete bipartite graph with m and n vertices respectively.

Theorem 1.2[6] (J. C. Bermond, C. Huang, A. Rosa and D. Sotteau, 1980) *There exists a* $(v, K_{2,3}, 1)$*-GD if and only if* $v \equiv 0,\ 1,\ 4,\ 9 \pmod{12}$, *for* $v \geq 5$ *and* $v \neq 9,\ 12$.

In [10], the existence of $(v, K_{2,3}, \lambda)$-*GD* for $\lambda > 1$ has been already researched. However, some existence results were not contained in [10], as $(12s+5, K_{2,3}, 6t+3)$-*GD* for $s \geq 1$ and any t. In §3, we discuss the existence of $(v, K_{2,3}, \lambda)$-*GD* for $\lambda > 1$ and complete the existence spectrum as follows.

Theorem 1.3 *There exist* $(v, K_{2,3}, \lambda)$*-GD if and only if*

(1) $v \equiv 0, 1, 4, 9 \pmod{12}$ *for any* λ, *except* $(v, \lambda) = (9, 1),\ (12, 1)$;

(2) $v \equiv 2,\ 11 \pmod{12}$ *and* $\lambda \equiv 0 \pmod 6$;

(3) $v \equiv 3,\ 6,\ 7,\ 10 \pmod{12}$ *and* $\lambda \equiv 0 \pmod 2$;

(4) $v \equiv 5,\ 8 \pmod{12}$ *and* $\lambda \equiv 0 \pmod 3$, *except* $(v, \lambda) \in \{(5, 6t+3):\ t \geq 0\}$.

In this paper, our main purpose is to determine the values $p(v, K_{2,3}, \lambda)$ and $c(v, K_{2,3}, \lambda)$ for any v and λ. The following theorems will be presented in the §2 and §4.

Theorem 1.4 *There exist* $(v, K_{2,3}, \lambda)$*-OPD and* $(v, K_{2,3}, \lambda)$*-OCD, for any positive integers* λ *and* v, $v \geq 5$, *with the exceptions of the following non-optimal cases:*

(1) $p(9, K_{2,3}, 1) = 5,\ c(9, K_{2,3}, 1) = 7,\ p(12, K_{2,3}, 1) = 10\ ,\ c(12, K_{2,3}, 1) = 12;$
(2) $p(6, K_{2,3}, 1) = 1,\ c(6, K_{2,3}, 1) = 4,\ p(8, K_{2,3}, 1) = 3,\ c(8, K_{2,3}, 1) = 6;$
(3) $p(10, K_{2,3}, 1) = 6,\ c(7, K_{2,3}, 1) = 6,\ p(11, K_{2,3}, 1) = 8;$
(4) $p(5, K_{2,3}, 6t+3) = 10t+4,\ p(5, K_{2,3}, 6t+5) = 10t+7,$
$c(5, K_{2,3}, 6t+1) = 10t+3,\ c(5, K_{2,3}, 6t+3) = 10t+6,\quad \text{for } t \geq 0.$

In what follows, we will denote $K_{2,3}$ by the notation $(a, b; c, d, e)$, where the vertex-set is $\{a, b, c, d, e\}$ and the edge-set is $\{(a, c),\ (a, d),\ (a, e),\ (b, c),\ (b, d),\ (b, e)\}$. In order to state more clearly, we will write down the corresponding leave-edge graph (repeat-edge graph) in each construction.

2. Case $\lambda = 1$

Lemma 2.1 *There exist $(v, K_{2,3}, 1)$-OPD for $v = 5$ and 7.*

Proof $(5, K_{2,3}, 1)$-*OPD* : $\mathcal{B} = \{(0, 1; 2, 3, 4)\},\ L_1 = \{(0, 1),\ (2, 3),\ (2, 4),\ (3, 4)\}.$
$(7, K_{2,3}, 1)$-*OPD* : $\mathcal{B} = \{(0, 1; 2, 3, 4),\ (3, 4; 2, 5, 6),\ (5, 6; 0, 1, 2)\},$
$L_1 = \{(0, 1), (3, 4), (5, 6)\}.$ □

Theorem 2.2 *There exist $(v, K_{2,3}, 1)$-OPD for $v \geq 14$.*

Proof Consider values of v according to their residue class mod 12. The classes $v \equiv 0, 1, 4, 9\ (mod\ 12)$ have been already covered by Theorem 1.2. For the other cases we give the constructions as follows.

(1) $v \equiv 2, 11\ (\text{mod } 12),\ v \geq 23$:

By Theorem 1.2, there is a $(v-2, K_{2,3}, 1)$-*GD*, say $(X, \mathcal{A})$. Let $\bigcup_{i=1}^{\frac{v-2}{3}} \{x_i, y_i, z_i\}$ be a partition of X and $\{a,\ b\} \bigcap X = \emptyset$. Define a collection of $K_{2,3}$'s: $\mathcal{B} = \{(a, b; x_i, y_i, z_i) : 1 \leq i \leq \frac{v-2}{3}\}$. Then $(X \bigcup \{a, b\}, \mathcal{A} \bigcup \mathcal{B})$ is a $(v, K_{2,3}, 1)$-*OPD* and $L_1 = \{(a, b)\}$.

(2) $v \equiv 3,\ 6\ (\text{mod } 12),\ v \geq 15$:

By Theorem 1.2, there is a $(v-2, K_{2,3}, 1)$-*GD*, say $(X, \mathcal{A})$. Let $\bigcup_{i=1}^{\frac{v-3}{3}} \{x_i, y_i, z_i\}$ be a partition of $X \backslash \{x_0\}$ for a given vertex $x_0 \in X$. Let $\{a,\ b\} \bigcap X = \emptyset$, we define a collection of $K_{2,3}$'s: $\mathcal{B} = \{(a, b; x_i, y_i, z_i) : 1 \leq i \leq \frac{v-3}{3}\}$. Then $(X \bigcup \{a, b\}, \mathcal{A} \bigcup \mathcal{B})$ is a $(v, K_{2,3}, 1)$-*OPD* and $L_1 = \{(a, b),\ (a, x_0),\ (b, x_0)\}$.

(3) $v \equiv 5, 8\ (\text{mod } 12),\ v \geq 17$:

By Theorem 1.2, there is a $(v-4, K_{2,3}, 1)$-*GD*, say $(X, \mathcal{A})$. Let $\bigcup_{i=1}^{\frac{v-5}{3}} \{x_i, y_i, z_i\}$ and $\bigcup_{i=1}^{\frac{v-5}{3}} \{x_i', y_i', z_i'\}$ be two partitions of $X \backslash \{x_0\}$ for a given vertex $x_0 \in X$, where two partitions can be identical. Let $\{a, b, c, d\} \bigcap X = \emptyset$ and $(\{x_0, a, b, c, d\},\ \mathcal{B}')$ be a $(5, K_{2,3}, 1)$-*OPD*. Define a collection of $K_{2,3}$'s

$$\mathcal{B} = \{(a, b; x_i, y_i, z_i),\ (c, d; x_i', y_i', z_i') : 1 \leq i \leq \tfrac{v-5}{3}\}.$$

Then $(X \bigcup \{a, b, c, d\}, \mathcal{A} \bigcup \mathcal{B}' \bigcup \mathcal{B})$ is a $(v, K_{2,3}, 1)$-*OPD* and its leave-edge graph is the same as that of $(5, K_{2,3}, 1)$-*OPD* on the vertex set $\{x_0, a, b, c, d\}$.

(4) $v \equiv 7, 10 \pmod{12}$, $v \geq 19$:

By Theorem 1.2, there is a $(v-6, K_{2,3}, 1)$-GD, say $(X, \mathcal{A})$. Let $\bigcup_{i=1}^{\frac{v-7}{3}}\{x_i,\ y_i,\ z_i\}$, $\bigcup_{i=1}^{\frac{v-7}{3}}\{x_i',\ y_i',\ z_i'\}$ and $\bigcup_{i=1}^{\frac{v-7}{3}}\{x_i'',\ y_i'',\ z_i''\}$ be three partitions of $X\backslash\{x_0\}$ for a given vertex $x_0 \in X$, where these partitions can be identical. Let $\{a,b,c,d,e,f\}\bigcap X = \emptyset$ and $(\{x_0,a,b,c,d,e,f\}, \mathcal{B}')$ be a $(7, K_{2,3}, 1)$-OPD. Define a collection of $K_{2,3}$'s

$$\mathcal{B} = \{(a,b;x_i,y_i,z_i),\ (c,d;x_i,y_i,z_i),\ (e,f;x_i,y_i,z_i) : 1 \leq i \leq \tfrac{v-7}{3}\}.$$

Then $(X\bigcup\{a,b,c,d,e,f\}, \mathcal{A}\bigcup\mathcal{B}'\bigcup\mathcal{B})$ is a $(v, K_{2,3}, 1)$-OPD and its leave-edge graph is the same as that of $(7, K_{2,3}, 1)$-OPD on the vertex set $\{x_0,a,b,c,d,e,f\}$.

(5) $(14, K_{2,3}, 1)$-OPD : $X = Z_{10}\bigcup\{x,y,z,t\}$,

$\mathcal{B}$: $(0,9;7,8,x)$, $(7,8;4,5,6)$, $(2,3;7,8,9)$, $(4,z;9,x,t)$, $(2,6;0,4,t)$,
$(1,x;2,7,8)$, $(4,5;0,1,3)$, $(2,4;5,y,z)$, $(7,8;y,z,t)$, $(5,t;9,x,y)$,
$(0,1;3,y,t)$, $(3,5;6,z,t)$, $(0,6;1,9,z)$, $(3,6;2,x,y)$, $(1,y;9,x,z)$.

$L_1 = \{(7,8)\}$. □

Theorem 2.3 *There exist $(v, K_{2,3}, 1)$-OCD for $v \geq 14$.*

Proof (1) $v \equiv 0,\ 1,\ 4,\ 9 \pmod{12}$: See Theorem 1.2.

(2) $v \equiv 2,\ 11 \pmod{12}$, $v \geq 23$:

Let $(Z_{v-2}\bigcup\{a,\ b\},\ \mathcal{A}\bigcup\mathcal{B})$ be the $(v,\ K_{2,3},\ 1)$-OPD given in Theorem 2.2(1), in which $L_1 = \{(a,\ b)\}$. Adding a block $(a,\ x;\ b,\ y,\ z)$ into $\mathcal{A}\bigcup\mathcal{B}$, we obtain just a $(v,\ K_{2,3},\ 1)$-OCD, where $x,\ y,\ z \in Z_{v-2}$ and $R_1 = \{(a,y),(a,z),(b,x),(x,y),(x,z)\}$.

(3) $v \equiv 3,\ 6\ (mod\ 12)$, $v \geq 15$:

Let $(Z_{v-2}\bigcup\{a,b\},\ \mathcal{A}\bigcup\mathcal{B})$ be the $(v, K_{2,3}, 1)$-OPD given in Theorem 2.2(2), in which $L_1 = \{(a,b),(a,x_0),(b,x_0)\}$. Without loss of generality, there is a block $(x_0,x;y,\ z,t) \in \mathcal{A}$, where $x,y,z,t \in Z_{v-2}$. Let $\bigcup_{i=1}^{\frac{v-3}{3}}\{x_i,y_i,z_i\}$ be the partition of $Z_{v-2}\backslash\{x_0\}$ given in Theorem 2.2(2). Noting that the arbitrariness of the partition, let $(x_1,y_1,z_1) = (x,y,z)$. Taking $H = \{(x_0,x;y,z,t),(a,b;x,y,z)\} \subset \mathcal{A}\bigcup\mathcal{B}$, let $H' = \{(x,x_0;a,b,t),\ (x,a;y,z,b),\ (b,x_0;y,z,t)\}$. Then, $(Z_{v-2}\bigcup\{a,\ b\},\ (\mathcal{A}\bigcup\mathcal{B}\bigcup H')\backslash H)$ is a $(v, K_{2,3}, 1)$-OCD and $R_1 = \{(x,b),(b,t),(t,x_0)\}$.

(4) $v \equiv 5,\ 8\ (mod\ 12)$, $v \geq 17$:

Let $(Z_{v-4}\bigcup\{a,b,c,d\}, \mathcal{A}\bigcup\mathcal{B}'\bigcup\mathcal{B})$ be the $(v, K_{2,3}, 1)$-OPD given in Theorem 2.2(3), in which $L_1 = \{(a,b),(a,c),(c,b),(x_0,d)\}$. Take a block $(x_0,t;x,y,z) \in \mathcal{A}$, where $x,y,z,t \in Z_{v-4}$. Let $\bigcup_{i=1}^{\frac{v-5}{3}}\{x_i,y_i,z_i\}$ and $\bigcup_{i=1}^{\frac{v-5}{3}}\{x_i',y_i',z_i'\}$ be the partitions of $Z_{v-4}\backslash\{x_0\}$ given in Theorem 2.2(3), where $(x_1,y_1,z_1) = (x,y,z)$ and $(x_1',y_1',z_1') = (y,z,t)$. Taking $H = \{(x_0,t;x,y,z),$ $(a,b;x,y,z),(c,d;y,z,t)\} \subset \mathcal{A}\bigcup\mathcal{B}'\bigcup\mathcal{B}$, let $H' = \{(x_0,b;x,y,z),(c,a;y,b,z),(d,t;x_0,y,z),(a,t;$ $c,d,x)\}$. Then, $(Z_{v-4}\bigcup\{a,b,c,d\}, (\mathcal{A}\bigcup\mathcal{B}\bigcup\mathcal{B}'\bigcup H')\backslash H)$ is a $(v, K_{2,3}, 1)$-OCD and $R_1 = \{(t,x_0),$ $(a,d)\}$.

(5) $v \equiv 7,\ 10 \pmod{12}$, $v \geq 19$:

Let $(Z_{v-6}\bigcup\{a,\ b,\ c,\ d,\ e,\ f\},\ \mathcal{A}\bigcup\mathcal{B}'\bigcup\mathcal{B})$ be the $(v,\ K_{2,3},\ 1)$-*OPD* given in Theorem 2.2(4), in which $L_1 = \{(d,\ e),\ (b,\ c),\ (a,\ x_0)\}$. Take a block $(x_0,\ t;\ x,\ y,\ z) \in \mathcal{A}$, where $x, y, z, t \in Z_{v-4}$. Let $\bigcup_{i=1}^{\frac{v-7}{3}}\{x_i, y_i, z_i\}$ and $\bigcup_{i=1}^{\frac{v-7}{3}}\{x_i', y_i', z_i'\}$ be the partitions of $Z_{v-6}\backslash\{x_0\}$ given in Theorem 2.2(4), where $(x_1, y_1, z_1) = (y, z, t)$ and $(x_1', y_1', z_1') = (x, y, t)$. Take $H = \{(x_0, t; x, y, z), (a, b; y, z, t), (c, d; x, y, t)\} \subset \mathcal{A}\bigcup\mathcal{B}'\bigcup\mathcal{B}$, and let

$$H' = \{(x_0, t; x, a, z), (c, d; e, x, t), (b, y; x_0, c, t), (y, z; a, b, d)\}.$$

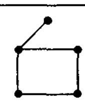
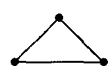
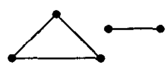
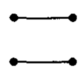
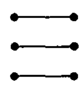
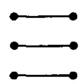
Then, $(Z_{v-6}\bigcup\{a, b, c, d, e, f\}, (\mathcal{A}\bigcup\mathcal{B}\bigcup\mathcal{B}'\bigcup H')\backslash H)$ is a $(v, K_{2,3}, 1)$-*OCD* and $R_1 = \{(b, x_0), (c, e), (d, z)\}$.

(6) $v = 14$:

A $(14,\ K_{2,3},\ 1)$-*OCD* can be formed by adding a block $(7,\ a;\ 8,\ b,\ c)$ to the $(14,\ K_{2,3},\ 1)$-*OPD* given in Theorem 2.2(5), where a, b and c are distinct points in $Z_{10}\bigcup\{x, y, z, t\}\backslash\{7, 8\}$ and $R_1 = \{(7, b),\ (7, c),\ (8, a),\ (a, b),\ (a, c)\}$. □

Below, we list the leave-edge graphs L_1 and repeat-edge graphs R_1 for each subcase. These graphs will play an important role in constructing *GD*, *OPD* and *OCD* for any λ.

Table A

$v \geq 14$	l_1	L_1	r_1	R_1
$\equiv 2,\ 11 \pmod{12}$	1		5	
$\equiv 3,\ 6 \pmod{12}$	3		3	
$\equiv 5,\ 8 \pmod{12}$	4		2	
$\equiv 7,\ 10 \pmod{12}$	3		3	

Suppose $H_1,\ H_2,\ \cdots,\ H_s$ be some subgraphs of K_v, where each $H_i = (x_i,\ y_i;\ a_i,\ b_i,\ c_i)$ is isomorphic to $K_{2,3}$, $1 \leq i \leq s$. Each a_i (or b_i or c_i) in any H_i is called a 2-*claw* in H_i, as well each x_i (or y_i) in any H_i is called a 3-*claw* in H_i. The union $\bigcup_{i=0}^{s} H_i$ is denoted by Ω_v. Let x be a vertex in Ω_v. The *degree-type* of x is denoted by $T(x) = 2^m3^n$ if x appears in m H_i (as 2-claw) and in n H_i (as 3-claw). Obviously, if the degree of x is denoted by $d(x)$, then $d(x) = 2m + 3n$. For example, if $\Omega_7 = H_1\bigcup H_2\bigcup H_3$ and

$$H_1 = \{(0, 1; 2, 3, 4)\},\quad H_2 = \{(3, 4; 2, 5, 6)\},\quad H_3 = \{(5, 6; 0, 1, 2)\},$$

then $T(0) = T(1) = T(3) = T(4) = T(5) = T(6) = 2^13^1$, $T(2) = 2^33^0 = 2^3$. Obviously, the subgraph family Ω_v is just the block set $\mathcal{B}$ (or $\mathcal{D}$) for $(v,\ K_{2,3},\ 1)$-*PD* (or $(v,\ K_{2,3},\ 1)$-*CD*). It is not difficult to verify the following properties.

Proposition 2.4 *Let* $\Omega_v=\bigcup_i H_i$ *and* $T(x)=2^{m_x}3^{n_x}$ *for* $x\in V(K_v)$.

(1) $\sum_x m_x=3|\Omega_v|,\ \sum_x n_x=2|\Omega_v|$;

(2) $n_x=n_y=0$ (or $m_x=m_y=0$) $\Longrightarrow$ edge (x,y) belongs to no $H_i(\in\Omega_v)$;

(3) $n_x=n_y=k$ (or $m_x=m_y=k$) and edge (x,y) belongs to some $H_i(\in\Omega_v)\Longrightarrow$ $|\{(x,\ y;\ \star,\ \star,\ \star)\in\Omega_v\}|<k$ (or $|\{(\star,\ \star;\ x,\ y,\ \star)\in\Omega_v\}|<k$);

(4) Let $Q=\{x\in V(K_v):\ T(x)=2^{m_x}3^1\}$ and

$$q=\begin{cases}|Q\bigcap V(L_1)| & \text{for packing } \mathcal{B}\\ 0 & \text{for covering } \mathcal{D},\end{cases}$$

then

$$|\Omega_v|-\sum_{n_x\geq 2} n_x\leq\lfloor\frac{q}{2}\rfloor. \tag{2.1}$$

For convenience, we list all the possible degree-type $T(x)$ for given $d(x)$, $5\leq d(x)\leq 13$.

$d(x)$	$T(x)$	$d(x)$	$T(x)$	$d(x)$	$T(x)$
5	2^13^1	8	$2^4,\ 2^13^2$	11	$2^13^3,\ 2^43^1$
6	$2^3,\ 3^2$	9	$2^33^1,\ 3^3$	12	$2^6,\ 3^4,\ 2^33^2$
7	2^23^1	10	$2^5,\ 2^23^2$	13	$2^53^1,\ 2^23^3$

Lemma 2.5[3] $p(9,K_{2,3},1)=5,\ c(9,K_{2,3},1)=7,\ p(12,K_{2,3},1)=10,\ c(12,K_{2,3},1)=12.$

Proof By Theorem 1.2, the following packing (covering) are maximum (minimum).

$(9,K_{2,3},1)$-*PD*, $\mathcal{B}:\ (0,1;2,5,7),\ (2,8;3,6,7),\ (3,6;0,1,7),\ (4,5;3,7,8),\ (4,8;0,1,2).$
$L_1=\{(0,\ 1),\ (2,\ 5),\ (3,\ 6),\ (4,\ 5),\ (4,\ 6),\ (5,\ 6)\}.$

$(9,K_{2,3},1)$-*CD*, $\mathcal{B}:\ (0,1;2,5,7),\ (2,8;3,6,7),\ (3,6;0,1,7),\ (4,5;3,7,8),$
$(4,8;0,1,2),\ (1,6;0,4,5),\ (5,6;2,3,4).$
$R_1=\{(0,\ 6),\ (2,\ 6),\ (4,\ 6),\ (1,\ 4),\ (1,\ 5),\ (3,\ 5)\}.$

$(12,K_{2,3},1)$-*PD*, $\mathcal{B}:\ (0,1;2,5,7),\ (4,8;0,1,2),\ (3,6;0,1,7),\ (4,5;3,7,8),\ (9,y;3,4,5),$
$(0,9;1,x,y),\ (2,6;5,9,x),\ (2,8;3,6,7),\ (7,8;9,x,y),\ (6,x;3,4,y).$
$L_1=\{(0,\ 9),\ (1,\ x),\ (1,\ y),\ (2,\ y),\ (4,\ 5),\ (5,\ x)\}.$

$(12,K_{2,3},1)$-*CD*, $\mathcal{B}:\ (0,1;2,5,7),\ (2,8;3,6,7),\ (6,x;3,4,y),\ (4,5;3,7,8),\ (3,6;0,1,7),$
$(4,8;0,1,2),\ (7,8;9,x,y),\ (9,y;3,4,5),\ (0,4;2,5,9),\ (2,6;5,9,x),$
$(0,9;1,x,y),\ (x,y;1,2,5).$
$R_1=\{(0,\ 2),\ (0,\ 5),\ (5,\ y),\ (2,\ x),\ (2,\ 4),\ (4,\ 9)\}.$ □

Lemma 2.6 $p(6,\ K_{2,3},\ 1)=1$ *and* $c(6,\ K_{2,3},\ 1)=4$.

Proof First, we have $p(6,\ K_{2,3},\ 1)\leq\lfloor\frac{6\times5}{12}\rfloor=2$ and $c(6,\ K_{2,3},\ 1)\geq\lceil\frac{6\times5}{12}\rceil=3$. It is easy to see that $H=K_6-K_{2,3}$ is a union of K_4 and K_3 with one common vertex. Obviously, there is no subgraph $K_{2,3}$ in H. Thereby, $p(6,\ K_{2,3},\ 1)=1$ and there exists no $(6,K_{2,3},1)$-*OPD*.

Furthermore, it is not difficult to see that the graph H can not be covered by two $K_{2,3}$'s. Thus, there is no $(6, K_{2,3}, 1)$-OCD. Here we give a minimum (6, $K_{2,3}$, 1)-CD :

$\mathcal{D}$: (0, 1; 2, 3, 4), (0, 5; 1, 2, 3), (2, 3; 1, 4, 5), (3, 5; 0, 2, 4).

$R_1 = \{(0, 2), (0, 3), (0, 3), (1, 2), (1, 3), (2, 5), (2, 5), (3, 4), (3, 5)\}$. □

Lemma 2.7 $c(5, K_{2,3}, 1) = 3$ *and* $c(7, K_{2,3}, 1) = 5$.

Proof (1) There exists no (5, $K_{2,3}$, 1)-OCD. In fact, $|\mathcal{D}| = c(5, K_{2,3}, 1) \geq \lceil\frac{5\times4}{12}\rceil = 2$. But $K_5 - K_{2,3}$ is a union of disjoint K_3 and K_2, which cannot occur in one $K_{2,3}$. So, there is no (5, $K_{2,3}$, 1)-OCD. Here, we give a minimum (5, $K_{2,3}$, 1)-CD :

$\mathcal{D}$: (0, 1; 2, 3, 4), (0, 2; 1, 3, 4), (2, 4; 0, 1, 3)..

$R_1 = \{(0, 2), (0, 3), (0, 4), (0, 4), (1, 2), (1, 2), (1,4), (2, 3)\}$.

(2) There exists no (7, $K_{2,3}$, 1)-OCD. Suppose there is a (7, $K_{2,3}$, 1)-OCD, say (X, $\mathcal{D}$). Then $s = \lceil\frac{7\times6}{12}\rceil = 4$ and $r_1 = 3$. By Proposition 2.4, we consider all possibilities of R_1 with 3 edges.

Case 1 If $K_7 \bigcup R_1$ has at least three vertices with degree 6 (there are five such graphs), then there exist at least two vertices with the same type 2^3 or 3^2 which is contradict to Proposition 2.4(2).

Case 2 If R_1 is a union of three disjoint P_2, then there are six vertices with degree 7 and one vertex with degree 6 in $K_7 \bigcup R_1$. Then we have :

$T(x)$	2^3	3^2	$2^2 3^1$
number of vertices x	m	$1-m$	6

By Proposition 2.4(1), $3m + 2 \times 6 = 3s = 12$ implies $m = 0$. Let $T(z) = 3^2$ for certain $z \in X$, then the other six vertices of X have the same degree-type $2^2 3^1$. It is not difficult to see that the structure of $\mathcal{D}$ must be in the form :

$$(z, \triangle; \star, \star, \star), (z, \triangle; \star, \star, \star), (\triangle, \triangle; \diamond, \diamond, \diamond), (\triangle, \triangle; \diamond, \diamond, \diamond),$$

where the six $\triangle$'s, the six $\star$'s and the six $\diamond$'s are all partitions of $X\backslash\{z\}$. Thus the last two blocks are contradict to Proposition 2.4(3).

Case 3 If R_1 is a union of disjoint P_2 and P_3, then there are two vertices with degree 6, four vertices with degree 7 and one vertex with degree 8 in $K_7 \bigcup R_1$. Then, we have

$T(x)$	2^3	3^2	$2^2 3^1$	2^4	$2^1 3^2$
number of vertices x	m	$2-m$	4	n	$1-n$

By Proposition 2.4(1), $3m + 2 \times 4 + 4n + 1 - n = 12$, i.e., $m + n = 1$. But, by Proposition 2.4(2), $m \neq 0$, thus $m = 1$ and $n = 0$. For $R_1 = P_2 \bigcup P_3$, let $P_2 = (a, b)$, $P_3 = (x, y, z)$ and the other vertices in K_7 are c and d. Then we have $T(y) = 2^1 3^2$, $T(a) = T(b) = T(x) = T(z) = 2^2 3^1$, $T(c) = 2^3$ and $T(d) = 3^2$. It is not difficult to see that the structure of $\mathcal{D}$ must be in the form : $(\triangle, \triangle; c, \star_1, \star_2)$, $(y, \triangle; c, \star_3, \star_4)$, $(d, \triangle; c, y, \diamond_1)$, $(d, y; \diamond_2, \diamond_3, \diamond_4)$, where $\{\diamond_1, \diamond_2, \diamond_3, \diamond_4\} = \{a, b, x, z\} = \{\star_1, \star_2, \star_3, \star_4\}$. Thereby, $\{\star_1, \star_2\} \subset \{\diamond_2, \diamond_3, \diamond_4\}$ or $\{\star_3, \star_4\} \subset \{\diamond_2, \diamond_3, \diamond_4\}$. It is impossible by Proposition 2.4(3) for $T(\star_j) = T(\diamond_i) = 2^2 3^1$ and $1 \le i, j \le 4$.

Therefore, there is no $(7, K_{2,3}, 1)$-OCD. The following $(7, K_{2,3}, 1)$-CD implies the conclusion $c(7, K_{2,3}, 1) = 5$:

$\mathcal{D}$: (0, 1; 2, 3, 4), (0, 3; 1, 4, 5), (1, 5; 2, 6, 4), (3, 4; 2, 5, 6), (5, 6; 0, 1, 2).

$R_1 = \{(0, 4), (0, 5), (1, 2), (1, 3), (1, 4), (1, 6), (2, 5), (3, 5), (4, 5)\}$. □

Lemma 2.8 *$p(8, K_{2,3}, 1) = 3$, $c(8, K_{2,3}, 1) = 6$, $p(10, K_{2,3}, 1) = 6$ and $p(11, K_{2,3}, 1) = 8$.*

Proof Similar to Lemma 2.7, we will give a detailed proof in Appendix, which is published in our website: http://qdkang.hebtu.edu.cn (online). Here we will give the maximum packings (or minimum covering) for these orders.

$(8, K_{2,3}, 1)$-PD, $\mathcal{B}$: $(3,4;0,1,2)$, $(6,7;0,1,2)$, $(6,7;3,4,5)$.

$L_1 = \{(0,1),(0,2),(0,5),(1,2),(1,5),(2,5),(3,4),(3,5),(4,5),(6,7)\}$.

$(8, K_{2,3}, 1)$-CD, $\mathcal{B}$: $(3,4;0,1,2)$, $(0,6;1,2,7)$, $(6,7;0,1,2)$,

$(6,7;3,4,5)$, $(2,5;0,1,3)$, $(3,5;1,2,4)$.

$R_1 = \{(0,2),(0,7),(1,3),(1,5),(1,6),(2,3),(2,3),(2,6)\}$.

$(10, K_{2,3}, 1)$-PD, $\mathcal{B}$: $(0,4;5,6,7)$, $(1,3;2,4,5)$, $(6,7;1,2,3)$, $(6,7;5,8,9)$,

$(8,9;0,4,5)$, $(8,9;1,2,3)$.

$L_1 = \{(0,1),(0,2),(0,3),(0,4),(1,3),(2,4),(2,5),(6,7),(8,9)\}$.

$(11, K_{2,3}, 1)$-PD $\mathcal{B}$: $(4,8;0,1,2)$, $(9,x;0,1,2)$, $(3,6;0,1,7)$, $(4,5;3,7,8)$,

$(0,1;2,5,7)$, $(9,x;3,4,5)$, $(2,8;3,6,7)$, $(9,x;6,7,8)$.

$L_1 = \{(0,1),(2,5),(3,6),(4,5),(4,6),(5,6),(9,x)\}$.. □

Lemma 2.9 *There exist $(v, K_{2,3}, 1)$-OCD for $v = 10$ and 11.*

Proof

$(10, K_{2,3}, 1)$-OCD, $\mathcal{B}$: $(0,6;2,3,5)$, $(1,7;0,6,8)$, $(1,8;0,3,4)$, $(4,9;0,5,6)$,

$(2,6;0,7,8)$, $(2,7;1,4,5)$, $(3,9;2,4,7)$, $(5,9;1,3,8)$.

$R_1 = \{(0, 1), (0, 2), (6, 7)\}$.

$(11, K_{2,3}, 1)$-OCD, $\mathcal{B}$: $(0,1;2,4,8)$, $(0,2;x,6,7)$, $(3,9;0,1,2)$, $(3,9;x,6,7)$,

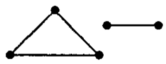
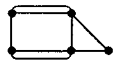
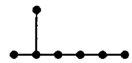
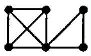
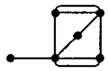
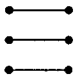
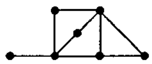
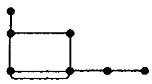
$(1,5;0,x,6)$, $(7,5;1,4,8)$, $(2,3;4,5,8)$, $(6,9;3,5,8)$,

$(4,x;6,8,9)$, $(4,7;5,6,x)$.

$R_1 = \{(4, 5), (4, 6), (5, 6), (3, 6), (9, x)\}$. □

Now, let us list the leave-edge graphs and the repeat-edge graphs for given maximal packing designs and minimal covering designs in our constructions, where $5 \le v \le 15$ and $\lambda = 1$.

Table B

v	L_1	R_1	v	L_1	R_1
5			9		
6			10		
7			11		
8			12		

3. Graph designs for $\lambda > 1$

The necessary condition to exist a $(v,\ K_{2,3},\ \lambda)$-GD is $\lambda v(v-1) \equiv 0$ $(mod\ 12)$. Let λ_{min} be the minimum positive integer λ satisfying this condition. Obviously, λ_{min} should be a factor of 6, and the existence of $(v,\ K_{2,3},\ \lambda_{min})$-$GD$ implies the existence of $(v,\ K_{2,3},\ n\lambda_{min})$-$GD$ for any positive integer n. We have the values of λ_{min} as follows.

$\lambda_{\min}$	1	2	3	6
$v \equiv$ (mod 12)	0, 1, 4, 9	3, 6, 7, 10	5, 8	2, 11

Theorem 3.1 *For* $v \equiv 0,\ 1,\ 4,\ 9$ (mod 12), $v \geq 5$ *and* $\lambda \geq 1$, *there exist* $(v,\ K_{2,3},\ \lambda)$-GD *with the exceptions of* $(v,\ \lambda) = (9,\ 1)$ *and* $(12,\ 1)$.

Proof For $v \equiv 0,\ 1,\ 4,\ 9$ (mod 12) and $v \neq 9$ and 12, there exists a $(v,\ K_{2,3},\ 1)$-GD by Theorem 1.2. Since $\lambda_{\min} = 1$ in this case, there exist $(v,\ K_{2,3},\ \lambda)$-GD for any positive integer λ. However, there is no $(v,\ K_{2,3},\ 1)$-GD for $v = 9$ and 12. But there exist the following designs:

$(9, K_{2,3}, 2)$-GD, $X = Z_3 \times Z_3$,
$\mathcal{B}: (0_0,\ 1_2;\ 0_1,\ 0_2,\ 1_1),\ (0_0,\ 0_2;\ 0_1,\ 1_0,\ 1_2),$
$(0_0,\ 2_1;\ 0_2,\ 1_1,\ 1_2),\ (1_0,\ 1_1;\ 0_1,\ 0_2,\ 2_0)$ mod $(3,\ -)$.

$(9, K_{2,3}, 3)$-GD, $X = Z_9$,
$\mathcal{B}: (0,\ 1;\ 2,\ 3,\ 4),\ (0,\ 2;\ 1,\ 4,\ 6)$ mod 9.

$(12, K_{2,3}, 2)$-GD, $X = Z_{11} \bigcup \{\infty\}$,
$\mathcal{B}: (0,\ 1;\ 2,\ 3,\ \infty),\ (0,\ 2;\ 3,\ 7,\ 6)$ mod 11.

$(12, K_{2,3}, 3)$-GD, $X = Z_{11} \bigcup \{\infty\}$,
$\mathcal{B}: (1,\ \infty;\ 0,\ 2,\ 3),\ (0,\ 2;\ 3,\ 7,\ 6),\ (0,\ 1;\ 3,\ 7,\ 9)$ mod 11.

Furthermore, for any positive integer $\lambda \geq 2$, there exist nonnegative integers s and t such that $\lambda = 2s + 3t$. Thus, there exist $(v,\ K_{2,3},\ \lambda)$-*GD*s for $v = 9,\ 12$ and $\lambda \geq 2$. □

Theorem 3.2 *There exist* $(v,\ K_{2,3},\ 6)$-*GD for* $v \equiv 2,\ 11 \pmod{12}$ *and* $v \geq 11$.

Proof By Theorem 2.2, there exists a $(v,\ K_{2,3},\ 1)$-*OPD* with $l_1 = 1$ for $v \equiv 2,\ 11 \pmod{12}$ and $v \geq 15$. Take six $(v,\ K_{2,3},\ 1)$-*OPD*'s on the same v-set X, say $\mathcal{B}_1,\ \mathcal{B}_2,\ \cdots,\ \mathcal{B}_6$. Let $a,\ b,\ c,\ x,\ y \in X$. Without loss of generality, the leave-edge graphs of these $\mathcal{B}_i$ can be chosen as $(a,\ x),\ (b,\ x),\ (c,\ x),\ (a,\ y),\ (b,\ y)$ and $(c,\ y)$ respectively. Then $(X,\ (\bigcup_{i=1}^{6}\mathcal{B}_i)\bigcup\{(x,\ y;\ a,\ b,\ c)\})$ is just a $(v,\ K_{2,3},\ 6)$-*GD*. For the remaining two orders, $v = 11$ and 14, we can give the following constructions immediately.

$(11, K_{2,3}, 6)$-*GD*, $X = Z_{11}$,

$\mathcal{B}: (0,\ 1;\ 2,\ 3,\ 4),\ (0,\ 1;\ 5,\ 6,\ 7),\ (3,\ 4;\ 0,\ 1,\ 2),$
$(1,\ 3;\ 0,\ 4,\ 5),\ (0,\ 6;\ 1,\ 4,\ 5) \mod 11.$

$(14, K_{2,3}, 6)$-*GD*, $X = Z_{13}\bigcup\{\infty\}$,

$\mathcal{B}: (0,\ \infty; 1,\ 2,\ 3),\ (\infty,\ 7;\ 1,\ 2,\ 3),\ (0,\ 7;\ 1,\ 2,\ 3) \times 5 \mod 13.$ □

Theorem 3.3 *There exist* $(v,\ K_{2,3},\ 2)$-*GD for* $v \equiv 3,\ 6,\ 7,\ 10 \pmod{12}$ *and* $v \geq 5$.

Proof By Theorem 2.2 and 2.3, when $v \equiv 7,\ 10 \pmod{12}$ and $v \geq 19$, there exist both $(v,\ K_{2,3},\ 1)$-*OPD*, say $(V,\ \mathcal{A})$, and $(v,\ K_{2,3},\ 1)$-*OCD*, say $(V,\ \mathcal{B})$. And, by Table A for the special structures given by us, the corresponding leave-edge graph L_1 and repeat-edge graph R_1 are isomorphic, i.e., both are three disjoint P_2's. Without loss of generality, we can let $L_1 = R_1$. Then, $(V, \mathcal{A}\bigcup\mathcal{B})$ is just a $(v,\ K_{2,3},\ 2)$-*GD*. For the remaining orders $v = 7$ and 10 in this case, we have:

$(7, K_{2,3}, 2)$-*GD*, $X = Z_7$, $\mathcal{B}:\ (0,\ 1;\ 2,\ 4,\ 6) \pmod 7$.

$(10, K_{2,3}, 2)$-*GD*, $X = Z_5 \times Z_2$,

$\mathcal{B}: (0_0, 4_0; 1_1, 3_1, 4_1),\ (0_0, 3_0; 1_0, 1_1, 2_0),\ (0_1, 3_1; 1_1, 2_1, 3_0) \mod (5,-).$

As for the cases $v \equiv 3$ or $6 \pmod{12}$, we give the following direct constructions.

$v \equiv 3 \pmod{12}$: $X = (Z_{6t+1} \times Z_2)\bigcup\{\infty\}$, $|\mathcal{B}| = (6t+1)(4t+1)$,

$$\mathcal{B}: \left.\begin{array}{l}(2_1,\ 3_1;\ \infty,\ 0_1,\ 1_1)\\(2_0,\ 3_0;\ \infty,\ 0_0,\ 1_0)\\(0_0,\ 2_0;\ 0_1,\ 3_1,\ 3_0)\\(0_1,\ 2_1;\ 0_0,\ 3_0,\ 3_1)\\(0_0,\ 4_0;\ 1_1,\ 2_1,\ 3_1)\end{array}\right\} \mod 6t+1.$$

$$\left.\begin{array}{l}(0_1,\ 0_0;\ (3i)_0,\ (3i-1)_0,\ (3i-2)_0) \times 2\\(0_1,\ 0_0;\ (3i)_1,\ (3i-1)_1,\ (3i-2)_1) \times 2\end{array}\right\} \mod 6t+1, \quad 2 \leq i \leq t.$$

$v \equiv 6 \pmod{12}$: $X = Z_{12t+5}\bigcup\{\infty\}$, $|\mathcal{B}| = (2t+1)(12t+5)$,

$\mathcal{B}:\ (6t+1,\ 6t+2;\ \infty,\ 0,\ 12t+3) \mod 12t+5,$
$(0,\ 3;\ 6i+4,\ 6i+5,\ 6i+6) \times 2 \mod 12t+5, \quad 0 \leq i \leq t-1.$ □

Theorem 3.4 *There exist* $(v, K_{2,3}, 3\lambda)$-*GD for positive integer* $v \equiv 5, 8 \pmod{12}$ *and* $\lambda > 0$ *with the exceptions of* $(v,\ \lambda) \in \{(5,\ 2t+1):\ t \geq 0\}$.

Proof (1) $v \equiv 5 \pmod{12}$ and $v \neq 5$, $X = Z_{12t+5}$, $|\mathcal{B}| = (3t+1)(12t+5)$,

$$\mathcal{B}: \left.\begin{array}{l}(0,\ 4;\ 6t-1,\ 6t,\ 6t+1)\\(0,\ 4;\ 6t,\ 6t+1,\ 6t+2)\\(0,\ 4;\ 6t-1,\ 6t,\ 6t+2)\\(0,\ 4;\ 6t-1,\ 6t+1,\ 6t+2)\end{array}\right\} \bmod 12t+5,$$

$$(0,\ 3;\ 6i+4,\ 6i+5,\ 6i+6)\times 3 \quad \bmod 12t+5,\ 0 \le i \le t-2.$$

(2) $v \equiv 8 \pmod{12}$, $X = Z_{12t+7} \bigcup \{\infty\}$, $|\mathcal{B}| = (3t+2)(12t+7)$,

$$\mathcal{B}: \left.\begin{array}{l}(\infty,\ 0;\ 6t+1,\ 6t+2,\ 6t+3)\\(0,\ 12t+4;\ 6t+1,\ 6t+2,\ 6t+3)\end{array}\right\} \bmod 12t+7,$$

$$(0,\ 3; 6i+4,\ 6i+5,\ 6i+6)\times 3 \quad \bmod 12t+7, \quad 0 \le i \le t-1.$$

(3) There exists a $(5,\ K_{2,3},\ 3\lambda)$-*GD* for even λ. It is enough to give a $(5,\ K_{2,3},\ 6)$-*GD* as follows : $(0,\ 2;\ 1,\ 3,\ 4)$ *and* $(0,\ 1;\ 2,\ 3,\ 4)$ *develop* 5.

(4) There exists no $(5,\ K_{2,3},\ 3\lambda)$-*GD* for odd λ. In fact, let the vertex set of K_5 be Z_5. All edges in K_5 are separated into two classes $\langle 1\rangle$ and $\langle 2\rangle$, where

$$\langle 1\rangle = \{(x,\ x+1):\ x \in Z_5\},\quad \langle 2\rangle = \{(x,\ x+2):\ x \in Z_5\}.$$

It is not difficult to see that, among six edges in any $K_{2,3}$ contained in K_5, there are four (or two) edges in the class $\langle 1\rangle$ and two (or four) edges in the class $\langle 2\rangle$. A $(5,\ K_{2,3},\ 3\lambda)$-*GD* consists of 5λ $K_{2,3}$'s, which cover exactly $3\lambda K_5$. It is impossible for odd λ, since the number of edges in difference class $\langle 1\rangle$ (or $\langle 2\rangle$) is even for 5λ $K_{2,3}$'s, but the number of edges in same class is odd for $3\lambda K_5$. □

Summarizing all the results of Theorems 3.1–3.4 and Theorem 1.2, the conclusion of Theorem 1.3 follows.

4. Packing and covering designs for $\lambda > 1$

The following Lemma is a modifying version of Theorem 4 in Section 3 of [11].

Lemma 4.1 *Given positive integers v, λ and μ. Let X be a v-set.*

(1) *Suppose there exist a $(v,\ K_{2,3},\ \lambda)$-OPD=$(X,\ \mathcal{D})$ with leave-edge graph $L_\lambda(\mathcal{D})$ and a $(v,\ K_{2,3},\ \mu)$-OPD=$(X,\ \mathcal{E})$ with leave-edge graph $L_\mu(\mathcal{E})$. If $|L_\lambda(\mathcal{D})| + |L_\mu(\mathcal{E})| = l_{\lambda+\mu}(v) < 6$, then there exists a $(v,\ K_{2,3},\ \lambda+\mu)$-OPD with leave-edge graph $L_\lambda(\mathcal{D}) \bigcup L_\mu(\mathcal{E})$.*

(2) Suppose there exist a $(v,\ K_{2,3},\ \lambda)$-*OCD*=$(X,\ \mathcal{D})$ with repeat-edge graph $R_\lambda(\mathcal{D})$ and a $(v,\ K_{2,3},\ \mu)$-*OCD*=$(X,\ \mathcal{E})$ with repeat-edge graph $R_\mu(\mathcal{E})$. If $|R_\lambda(\mathcal{D})| + |R_\mu(\mathcal{E})| = r_{\lambda+\mu}(v) < 6$, then there exists a $(v,\ K_{2,3},\ \lambda+\mu)$-*OCD* with repeat-edge graph $R_\lambda(\mathcal{D}) \bigcup R_\mu(\mathcal{E})$.

(3) Suppose there exist a $(v,\ K_{2,3},\ \lambda)$-*PD*=$(X,\ \mathcal{D})$ with leave-edge graph $L_\lambda(\mathcal{D})$ and a $(v,\ K_{2,3},\ \mu)$-*CD*=$(X,\ \mathcal{E})$ with repeat-edge graph $R_\mu(\mathcal{E})$. If $R_\mu(\mathcal{E}) \subset L_\lambda(\mathcal{D})$ and $|L_\lambda(\mathcal{D})| - |R_\mu(\mathcal{E})| = l_{\lambda+\mu}(v) < 6$, then there exists a $(v,\ K_{2,3},\ \lambda+\mu)$-*OPD* with leave-edge graph $L_\lambda(\mathcal{D}) \backslash R_\mu(\mathcal{E})$.

(4) Suppose there exist a $(v,\ K_{2,3},\ \lambda)$-*CD*=$(X,\ \mathcal{D})$ with repeat-edge graph $R_\lambda(\mathcal{D})$ and a $(v,\ K_{2,3},\ \mu)$-*PD*=$(X,\ \mathcal{E})$ with leave-edge graph $L_\mu(\mathcal{E})$. If $L_\mu(\mathcal{E}) \subset R_\lambda(\mathcal{D})$ and $|R_\lambda(\mathcal{D})| - |L_\mu(\mathcal{E})| = r_{\lambda+\mu}(v) < 6$, then there exists a $(v,\ K_{2,3},\ \lambda+\mu)$-*OCD* with repeat-edge graph

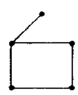
$R_\lambda(\mathcal{D})\backslash L_\mu(\mathcal{E})$.

In order to prove Theorem 1.4, for each v, we need only to consider the cases $1 < \lambda < \lambda_{min}$, where λ_{min} is the smallest λ to exist $(v,\ K_{2,3},\ \lambda)$-GD. However, for the case that there exists no $(v,\ K_{2,3},\ 1)$-OPD or $(v,\ K_{2,3},\ 1)$-OCD, we have yet to consider the additional case $\lambda = \lambda_{\min}+1$. Below, in the proof of Theorems 4.2-4.4 we will use the method given by Lemma 4.1 and the graphs listed in Table A and Table B.

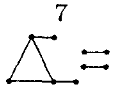
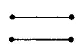
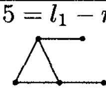
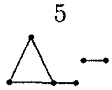
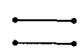
Theorem 4.2 *There exist $(v,\ K_{2,3},\ \lambda)$-OPD and $(v,\ K_{2,3},\ \lambda)$-OCD for $\lambda > 1$ and $v \equiv 2,\ 11 \pmod{12}$.*

Proof Here, $\lambda_{\min} = 6$.

For $v \geq 14$, by Theorem 2.2 and Theorem 2.3, there exist $(v,\ K_{2,3},\ 1)$-OPD and $(v,\ K_{2,3},\ 1)$-OCD. From the leave-edge graph L_1 and the repeat-edge graph R_1 listed in Table A and by Lemma 4.1, we can list the following table to get $(v,\ K_{2,3},\ \lambda)$-OPD and $(v,\ K_{2,3},\ \lambda)$-OCD for $1 < \lambda < 6$.

λ	1	2	3	4	5
l_λ	1	$2 = 2l_1$	$3 = l_1 + l_2$	$4 = 2l_2$	$5 = l_1 + l_4$
L_λ					
r_λ	5	$4 = r_1 - l_1$	$3 = r_2 - l_1$	$2 = r_3 - l_1$	$1 = r_4 - l_1$
R_λ					

For $v = 11$, there exists no $(v,\ K_{2,3},\ 1)$-OPD. From Table B, we have the table below, where $(11,\ K_{2,3},\ 2)$-OCD can be obtained from $(11,\ K_{2,3},\ 2)$-OPD by adding a block containing its leave-edges.

λ	1	2	3	4	5	7
l_λ	7	$2 = l_1 - r_1$	$3 = l_1 - r_2$	$4 = 2l_2$	$5 = l_1 - r_4$	$1 = l_2 - r_5$
L_λ						
r_λ	5	4	$3 = r_1 - l_2$	$2 = r_2 - l_2$	$1 = r_3 - l_2$	$5 = r_5 + r_2$
R_λ						

Theorem 4.3 *There exist $(v,\ K_{2,3},\ \lambda)$-OPD and $(v,\ K_{2,3},\ \lambda)$-OCD for $v \equiv 3,\ 6,\ 7, 10 \pmod{12}$ and $\lambda > 1$.*

Proof Here, $\lambda_{\min} = 2$.

For $v \geq 15$, by Theorem 2.2 and Theorem 2.3, there exist $(v,\ K_{2,3},\ 1)$-OPD and $(v,\ K_{2,3},\ 1)$-OCD. Since $\lambda_{\min} = 2$, we can get the desired conclusion immediately.

For $v = 6,\ 7$ and 10, there exists no $(v,\ K_{2,3},\ 1)$-OPD or $(v,\ K_{2,3},\ 1)$-OCD by Lemma 2.6-2.8. We need to construct $(v,\ K_{2,3},\ 3)$-OPD and $(v,\ K_{2,3},\ 3)$-OCD for these values of v.

First, for $v=6$, we give:

$$(6, K_{2,3}, 3)\text{-}OPD,\ \mathcal{B}:\ (0,1;2,3,4),\ (0,4;1,2,3),\ (0,2;3,4,5),\ (2,5;1,3,4),\ (1,5;0,2,3),\ (1,2;0,3,5),\ (4,5;0,1,3).$$
$$L_1=\{(4,\ 5),\ (4,\ 5),\ (3,\ 4)\}.$$
$$(6, K_{2,3}, 3)\text{-}OCD,\ \mathcal{B}:\ (0,1;2,3,4),\ (0,4;1,3,5),\ (0,2;3,4,5),\ (2,5;1,3,4),\ (1,5;2,3,4),\ (1,2;0,4,5),\ (4,5;0,1,3),\ (0,3;1,2,4).$$
$$L_1=\{(1,\ 4),\ (1,\ 4),\ (0,\ 4)\}.$$

For $v=7$, take the construction of $(v,\ K_{2,3},\ 1)$-*OPD* in Lemma 2.1 as $\mathcal{B}_1$. Let $\mathcal{B}_2=\sigma(\mathcal{B}_1)$ and $\mathcal{B}_3=\sigma^2(\mathcal{B}_2)$, where $\sigma=(0)(2)(3)(5)(146)$ is a transform on Z_7. Then, $\mathcal{B}=\mathcal{B}_1\bigcup\mathcal{B}_2\bigcup\mathcal{B}_3\bigcup\{(0,\ 3;\ 1,\ 4,\ 6)\}$ forms a $(7,\ K_{2,3},\ 3)$-*OPD*. In fact, $L_1(\mathcal{B}_1)=\{(0,\ 1),\ (3,\ 4),\ (5,\ 6)\}$, so $L_1(\mathcal{B}_2)=\{(0,\ 4),\ (3,\ 6),\ (1,\ 5)\}$ and $L_1(\mathcal{B}_3)=\{(0,\ 6),\ (1,\ 3),\ (4,\ 5)\}$. Thus, $L_3(\mathcal{B})=\{(1,\ 5),\ (4,\ 5),\ (5,\ 6)\}$. Furthermore, $\mathcal{A}=\mathcal{B}\bigcup\{(0,\ 5;\ 1,\ 4,\ 6)\}$ forms a $(7,\ K_{2,3},\ 3)$-*OCD*.

For $v=10$, take the following constructions in Lemma 2.9.

$(10,\ K_{2,3},\ 1)$-*PD* $=(Z_{10},\ \mathcal{B}_1)$, where $L_1(\mathcal{B}_1)=\{(0,\ 1),\ (0,\ 2),\ (6,\ 7),\ (0,\ 3),\ (0,\ 4),\ (8,\ 9),\ (1,\ 3),\ (2,\ 4),\ (2,\ 5)\}$; $(10,\ K_{2,3},\ 1)$-*OCD* $=(Z_{10},\ \mathcal{A}_1)$, where $R_1(\mathcal{A}_1)=\{(0,\ 1),\ (0,\ 2),\ (6,\ 7)\}$. Let $\mathcal{A}_2=\tau(\mathcal{A}_1)$, where $\tau=(0)(5)(13)(24)(68)(79)$ is a transformation on Z_{10}. Then, it is not difficult to see that $\mathcal{B}=\mathcal{B}_1\bigcup\mathcal{A}_1\bigcup\mathcal{A}_2$ and $\mathcal{A}=\mathcal{B}\bigcup\{(1,\ 2;\ 3,\ 4,\ 5)\}$ are $(10,\ K_{2,3},\ 3)$-*OPD* and $(10,\ K_{2,3},\ 3)$-*OCD* respectively. As well, $L_3(\mathcal{B})=\{(1,\ 3),\ (2,\ 4),\ (2,\ 5)\}$ and $R_3(\mathcal{A})=\{(2,\ 3),\ (1,\ 4),\ (1,\ 5)\}$. □

Theorem 4.4 *When* $v\equiv 5,\ 8 \pmod{12}$,

(1) there exist $(v,\ K_{2,3},\ \lambda)$-*OPD for* $(v,\ \lambda)\neq(5,\ 6t+5),\ (5,\ 6t+3),\ t\geq 0$;

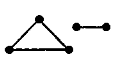
(2) there exist $(v,\ K_{2,3},\ \lambda)$-*OCD for* $(v,\ \lambda)\neq(5,\ 6t+1),\ (5,\ 6t+3),\ t\geq 0$;

(3) $p(5,K_{2,3},6t+3)=10t+4,\ \ p(5,K_{2,3},6t+5)=10t+7,$
$c(5,K_{2,3},6t+1)=10t+3,\ \ c(5,K_{2,3},6t+3)=10t+6.$

Proof Here, $\lambda_{\min}=3$.

Case 1 $(v\geq 17)$

By Theorem 2.2 and Theorem 2.3, there exist $(v,\ K_{2,3},\ 1)$-*OPD* and $(v,\ K_{2,3},\ 1)$-*OCD*. From the leave-edge graph L_1 and the repeat-edge graph R_1 listed in Table A and by Lemma 4.1, we can list the following table to get $(v,\ K_{2,3},\ \lambda)$-*OPD* and $(v,\ K_{2,3},\ \lambda)$-*OCD* for $1<\lambda<3$.

λ	1	2
l_λ	4	$2=l_1-r_1$
L_λ		
r_λ	2	$4=r_1+r_1$
R_λ		

Case 2 $(v=5)$

(1) There exist no $(5,\ K_{2,3},\ 1)$-*OCD* and no $(5,\ K_{2,3},\ 3)$-*GD* by Lemma 2.8 and Theorem 3.4. And, a $(5,\ K_{2,3},\ 2)$-*OPD* can be given as follows.

$\mathcal{B}$: (0, 1; 2, 3, 4), (0, 2; 1, 3, 4), (2, 1; 0, 3, 4). $L_2 = \{(3, 4), (3, 4)\}$.

(2) Suppose there exists a $(5,\ K_{2,3},\ 6t-1)$-*OPD* (or $(5,\ K_{2,3},\ 6t+1)$-*OCD*), then there are $10t-2$ (or $10t+2$) blocks in its block set $\mathcal{B}$ with 2 left (or repeated) edges. Obviously, every vertex should appear in each block. Let $x \in V(K_5)$, $T(x) = 2^p3^q$. By the equations listed in Proposition 2.4(1), we have $(p,\ q) = (6t-2+\lambda,\ 4t-\lambda)$ (or $(6t+2-\lambda,\ 4t+\lambda)$), where $0 \le \lambda \le 2$. Hence, the degree type of each vertex can only be

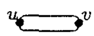
$$A.\ 2^{6t-2}3^{4t}(\text{or } 2^{6t+2}3^{4t}),\quad B.\ 2^{6t-1}3^{4t-1}(\text{or } 2^{6t+1}3^{4t+1}),\quad \text{or}\quad C.\ 2^{6t}3^{4t-2}(\text{or } 2^{6t}3^{4t+2}).$$

Suppose there are a vertices of type A, b vertices of type B, c vertices of type C. By the equations listed in Proposition 2.4(1), we get three solutions:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix},\ \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix},\ \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}.$$

So, there are only three possible structures (suppose $V(K_5) = \{x,\ y,\ z,\ u,\ v\}$)

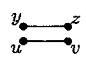
(i) $T(x) = T(y) = T(z) = 2^{6t-2}3^{4t}$; $T(u) = T(v) = 2^{6t}3^{4t-2}$.
(or $T(x) = T(y) = T(z) = 2^{6t+2}3^{4t}$; $T(u) = T(v) = 2^{6t}3^{4t+2}$.)

L_{6t-1} (or R_{6t+1}) : 

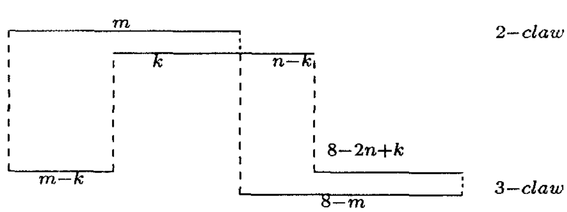
(ii) $T(x) = T(y) = 2^{6t-2}3^{4t}$; $T(u) = T(v) = 2^{6t-1}3^{4t-1}$; $T(z) = 2^{6t}3^{4t-2}$.
(or $T(x) = T(y) = 2^{6t+2}3^{4t}$; $T(u) = T(v) = 2^{6t+1}3^{4t+1}$; $T(z) = 2^{6t}3^{4t+2}$.)

L_{6t-1} (or R_{6t+1}) : 

(iii) $T(x) = 2^{6t-2}3^{4t}$; $T(y) = T(z) = T(u) = T(v) = 2^{6t-1}3^{4t-1}$.
(or $T(x) = 2^{6t+2}3^{4t}$; $T(y) = T(z) = T(u) = T(v) = 2^{6t+1}3^{4t+1}$.)

L_{6t-1} (or R_{6t+1}) : 

Let $s,\ t \in V(K_5)$, $T(s) = 2^m3^{8-m}$, $T(t) = 2^n3^{8-n}$. It is not difficult to see that the edge $(s,\ t)$ appears $m+n-2k$ times in $\mathcal{B}$:



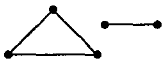
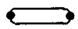
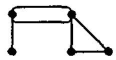
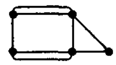
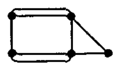
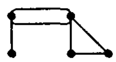
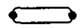
Obviously, the edge $(s,\ t)$ appears even times if $m = n$. In fact, $(u,\ v)$ need to appear $6t-3$ (or $6t+3$) times in (I), $(x,\ y)$ need to appear $6t-1$ (or $6t+1$) times in (II), and $(s,\ t)$ need to appear $6t-1$ (or $6t+1$)times in (III). It is contradict to the result given above. Thus, there exists no $(5,\ K_{2,3},\ 6t-1)$-*OPD* and no $(5,\ K_{2,3},\ 6t+1)$-*OCD*.

(3) First, we give a maximum $(5,\ K_{2,3},\ 5)$-*PD* as follows.

$\mathcal{B}$: (1, 3; 0, 2, 4) (mod 5), *and* (1, 2; 0, 3, 4), (2, 3; 0, 1, 4).

$L_5 = \{(0, 2), (0, 3), (0, 3), (0, 4), (1, 3), (1, 4), (1, 4), (2, 4)\}$.

Furthermore, by the existence of (5, $K_{2,3}$, 1)-*OPD* and (5, $K_{2,3}$, 6)-*GD*, we have the table as follows.

λ	1	2	3	4	5
l_λ	4	2	$6=l_1+l_2$	$4=l_2+l_2$	8
L_λ					
r_λ	8	$4=r_1-l_1$	$6=r_1-l_2$	$2=r_2-l_2$	$4=r_3-l_2$
R_λ					

There exist (5, $K_{2,3}$, λ)-*OPD* for $\lambda \not\equiv 3, 5 \pmod 6$; there exist (5, $K_{2,3}$, λ)-*OCD* for $\lambda \not\equiv 1, 3 \pmod 6$. Obviously,

$$p(5,K_{2,3},6t+3)=\frac{(6t+3)\times5\times4-6}{12}=10t+4,\quad p(5,K_{2,3},6t+5)=\frac{\frac{(6t+5)\times5\times4}{2}-8}{6}=10t+7;$$

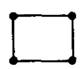
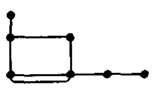
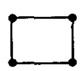
$$c(5,K_{2,3},6t+1)=\frac{\frac{(6t+1)\times5\times4}{2}+8}{6}=10t+3,\quad c(5,K_{2,3},6t+3)=\frac{(6t+3)\times5\times4+6}{12}=10t+6.$$

Case 3 ($v=8$)

There exist no (v, $K_{2,3}$, 1)-*OCD* and (v, $K_{2,3}$, 3)-*GD* by Lemma 2.8 and Theorem 3.4. We give the following constructions :

A (8, $K_{2,3}$, 2)-*OPD* : $\mathcal{B}$: (0, 1; 2, 3, 4), (0, 2; 3, 4, 5), (0, 1; 5, 6, 7),
(0, 5; 1, 6, 7), (1, 2; 0, 3, 7), (1, 7; 2, 4, 6),
(2, 3; 4, 5, 6), (3, 4; 5, 6, 7), (4, 7; 3, 5, 6).
$L_2=\{(2, 6), (5, 6)\}$.

A (8, $K_{2,3}$, 2)-*OCD* : $\mathcal{A}=\mathcal{B}\bigcup\{(2, 5; 1, 3, 6)\}$. $R_1=\{(1, 2), (1,5), (2, 3), (3, 5)\}$.
Furthermore, we have the table as follows.

λ	1	2	4
l_λ	10	2	$4=l_2+l_2$
L_λ			
r_λ	8	4	$2=r_2-l_2$
R_λ			

□

Summarizing the Theorems 4.2–4.4, we complete Theorem 1.4.

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λK_v 的最大 $K_{2,3}$ 填充设计和最小 $K_{2,3}$ 覆盖设计

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摘要: 对于一个有限简单图 G, λK_v 的 G- 设计 (G- 填充， G- 覆盖)，记为 (v,G,λ)-GD ((v,G,λ)-PD, (v,G,λ)-CD)，是一个 $(X,\mathcal{B})$，其中 X 是 K_v 的顶点集， $\mathcal{B}$ 是 K_v 的子图族，每个子图 (称为区组) 均同构于 G, 且 K_v 中任一边都恰好 (最多，至少) 出现在 $\mathcal{B}$ 的 λ 个区组中. 一个填充 (覆盖) 设计称为是最大 (最小) 的，如果没有其它的这种填充 (覆盖) 设计具有更多 (更少) 的区组. 本文对于 $\lambda > 1$ 确定了 $(v,K_{2,3},\lambda)$-GD 的存在谱，并对任意 λ 构造了 λK_v 的最大 $K_{2,3}$- 填充设计和最小 $K_{2,3}$- 覆盖设计.

关键词: G- 图设计； G- 填充设计； G- 覆盖设计.