第25卷第1期 数学研究与评论 Vol.25, No.2
2005年2月 JOURNAL OF MATHEMATICAL RESEARCH AND EXPOSITION Feb., 2005

Article ID: 1000-341X(2005)01-0058-06 **Document code:** A

3-Restricted Edge Connectivity of Vertex Transitive Graphs of Girth Three

OU Jian-ping[1,2], *ZHANG Fu-ji*[3]

(1. Dept. Math. Phys., Wuyi University, Jiangmen 529020, China;
2. Dept. Math., Zhangzhou Normal College, Fujian 363000, China;
3. Dept. Math., Xiamen University, Fujian 361005, China)
(E-mail: oujp@eyou.com)

Abstract: Let G be a k-regular connected graph of order at least six. If G has girth three, its 3-restricted edge connectivity $\lambda_3(G) \leq 3k-6$. The equality holds when G is a cubic or 4-regular connected vertex-transitive graph with the only exception that G is a 4-regular graph with $\lambda_3(G)=4$. Furthermore, $\lambda_3(G)=4$ if and only if G contains K_4 as its subgraph.

Key words: vertex-transitive graph; 3-restricted edge connectivity; restricted fragment.
MSC(2000): 05C40
CLC number: O157.5

1. Introduction

When studying network reliability, one often considers such a kind of model whose nodes never fail but edges fail independently of each other with equal probability. Traditional tools in dealing with this problem are edge connectivity and restricted edge connectivity, refer to [1-4] But these tools fail to work when comparing reliability of networks with optimal edge connectivity and restricted edge connectivity. To settle this problem, we employ the concept m-restricted edge connectivity.

Definition 1.1 *An m-restricted edge, or simply R_m-edge cut, is an edge cut of a connected graph which disconnects this graph with each component having order at least m. The size of a minimum R_m-edge cut of graph G is called its R_m-edge connectivity.*

Denoted by $\lambda_m(G)$ the m-restricted edge connectivity of graph G. Clearly, 1-restricted edge connectivity is the traditional edge connectivity, and 2-restricted is the so-called restricted edge connectivity. R_m-edge connectivity proves to be a more precise measure for the fault-tolerance and reliability of networks than some traditional measures such as edge connectivity. With the properties of 3-restricted edge connectivity, Meng[5] and Wang[6] gave an approximate expression of the reliability of networks with topology being vertex transitive graphs of girth at least 5, their observations implied that networks with greater 3-restricted edge connectivity have greater reliability in a sense. In [7] we considered the 3-restricted edge connectivity of vertex-

Received date: 2002-09-29
Foundation item: National Natural Science Foundation of China (10271105); Foundation of Education Ministry of Fujian (JA03147); Ministry of Science and Technology of Fujian (2003J036).

transitive graphs of girth four, but for the cases of girth three it seems much more difficult. In this paper we study the R_3-edge connectivity of vertex-transitive graphs of girth three that have low degree.

Let F be a subset of $V(G)$ or a subgraph of G. Denote by $G \setminus F$ the graph obtained from G by removing the vertices of F from G, simplify $G \setminus \{u\}$ as $G \setminus u$. For two disjoint subsets A and B of $V(G)$, or two vertex-disjoint subgraphs of G, $[A, B]$ indicates the set of edges with one end in A and the other in B. We simplify $[A, G \setminus A]$ as $I(A)$ and write $\partial(A)$ for $|I(A)|$. Let $\xi_3(G) = \min\{\partial(F) : F \text{ is a connected vertex induced subgraph of order 3 of } G\}$. It is proved that $\lambda_3(G) \leq \xi_3(G)$, graph G is called maximal R_3-edge connected if $\lambda_3(G) = \xi_3(G)$. A k-regular graph G is called vertex transitive if, for any two vertices u and v, there is an automorphism τ in $\mathrm{Aut}(G)$ such that $\tau(u) = v$, where $\mathrm{Aut}(G)$ is the automorphism group of graph G. Our main results are as follows

Theorem 1.1 *A 4-regular connected vertex transitive graph G of girth three and order at least six either is maximal R_3-edge connected or has $\lambda_3(G) = 4$. Moreover, $\lambda_3(G) = 4$ if and only if G contains K_4.*

Theorem 1.2 *Every cubic connected vertex transitive graph of girth three and order at least six is maximal R_3-edge connected.*

Before proceeding, we introduce some more notations and terminologies. For any minimum R_3-edge cut S, $G - S$ contains exactly two components, both are called the R_3-fragments, or simply fragments, corresponding to S, among which the smaller one (with less vertices) is called a normal fragment. If denote by X one of these two fragments, the other is denoted by X^c. We signify a fragment with its vertex set, fragments with minimum cardinality are called atoms. Denote by $\nu(G)$ and $\varepsilon(G)$ the order and size of graph G respectively. The length of the shortest cycle of graph G, denoted by $g(G)$, is called its girth, while the longest one, denoted by $c(G)$, is called its circumference. Write $\lambda(G)$ for the edge connectivity of graph G. For other symbols and terminologies not explicitly stated, we follow [8].

2. Auxiliary lemmas

Lemma 2.1[4] *Let G be a connected graph with order not less than $2m$. If $c(G) \geq m + 1$, then G contains R_m-edge cuts.*

Lemma 2.2[9] *A k-regular connected vertex transitive graph is k edge connected.*

Lemma 2.3 *Let G be a connected vertex transitive k-regular graph of girth 3 and order at least 6. If $k \geq 3$, then $\lambda(G) \leq \lambda_2(G) \leq \lambda_3(G) \leq \xi_3(G) = 3k - 6$.*

Proof Since a k-regular graph has a cycle of length at least $k + 1$ if $k \geq 2$, by Lemma 2.1 it follows immediately that this kind of graph contains R_3-edge cut. Note that every R_3-edge cut is an R_2-edge cut, and that every R_2-edge cut is an edge cut, the first two inequalities of Lemma 2.3 is thus true. Let T be a triangle of G and H be a component of $G \backslash T$, since $k \geq 3$ and $|G| \geq 6$ it follows that H is neither an isolated vertex nor an edge. Hence the edges between H and T

form an R_3-edge cut with size $\partial(T) = \xi_3(G)$. Therefore $\lambda_3(G) \le \partial(T) = \xi_3(G) = 3k - 6$ as desired.

In the rest of this paper we restrict ourselves to 4-regular connected vertex-transitive graphs of girth 3 and order at least 6, any such graph G has $\xi_3(G) = 6$. Let X be a connected subgraph of order 4 and girth 3, then the possible value of $\partial(X)$ is $8, 6$ and 4. If $\lambda_3(G) = 5$, then any atom of G has order at least 5. Note that this trivial result is very useful in the following discussion.

Lemma 2.4 *Let X and Y be two different fragments of graph G. If the following two conditions hold,*

(1) $\nu(X \cap Y) \ge 3$; (2) $\partial(X \cap Y) \le \lambda_3(G)$,

then $X \cap Y$ is a fragment.

Proof Let X and X^c be the two fragment corresponding to an R_3-edge cut S, Y and Y^c be the two fragments corresponding to an another R_3-edge cut T. Let

$$A = X \cap Y,\ B = X \cap Y^c,\ C = X^c \cap Y,\ D = X^c \cap Y^c.$$

It suffices to show that both A and A^c are connected.

Claim 1. A^c is connected.

Noting that both X^c and Y^c are connected fragments of G, we conclude that A^c is connected if D is not empty. When D is empty, if $[B, C] = \emptyset$ then $S = [A, C], T = [A, B]$ and $I(A) = S \cup T$. It follows that

$$\lambda_3(G) \ge \partial(A) = \partial(S) + \partial(T) = 2\lambda_3(G) > \lambda_3(G).$$

This contradiction implies that if $D = \emptyset$ then $[B, C] \ne \emptyset$, and thus A^c is connected in either case. Claim 1 follows.

Claim 2. A is connected.

Suppose, to the contrary, that A is not connected with components $A_i, i = 1, 2, ..., m, m \ge 2$. By Lemma 2.2, we have $\lambda(G) = 4$. Therefore

$$\lambda_3(G) \ge \partial(A) = \sum_{i=1}^{m} \partial(A_i) \ge 2\lambda(G) = 8 > 6 = \xi_3(G) \ge \lambda_3(G).$$

This is a contradiction to condition (2), Claim 2 follows.

Lemma 2.5 *Let X and Y be two distinct normal fragments of G. If $\nu(X \cap Y) \ge 3$, then $X \cap Y$ is a fragment.*

Proof According to Lemma 2.4, it suffices to prove $\partial(X \cap Y) \le \lambda_3(G)$. Let X and Y be the two fragments corresponding to R_3-edge cut S and T respectively. Define A, B, C and D as in the proof of Lemma 2.4. Since

$$\nu(A) + \nu(B) = \nu(X) \le \nu(G)/2 \le \nu(Y^c) = \nu(B) + \nu(D),$$

it follows that $\nu(D) \ge \nu(A)$.

Claim 1. $\partial(D) \geq \lambda_3(G)$.

This claim is obviously true when D is connected since in this case $I(D)$ is an R_3-edge cut. If D is not connected with components $D_i, i = 1, 2, ..., p, p \geq 2$, according to Lemma 2.2, we have $\partial(D_i) \geq k = 4$. Therefore

$$\partial(D) \geq \partial(D_1) + \partial(D_2) \geq 2k = 8 > 6 = \xi_3(G) \geq \lambda_3(G).$$

Claim 1 is thus true in either case.

Combining Claim 1 and the following formula

$$\begin{aligned}\partial(A) + \partial(D) =& |[A,B]| + |[A,C]| + 2|[A,D]| + |[B,D]| + |[D,C]| + 2|[B,C]| - 2|[B,C]| \\ =& |S| + |T| - 2|[B,C] \leq |S| + |T| = 2\lambda_3(G),\end{aligned}$$

we have $\partial(A) \leq \lambda_3(G)$ as desired.

Since atoms are normal fragments that contain no fragments as their proper subgraphs (with less vertices), according to Lemma 2.5, we can easily prove the following

Corollary 2.6 *If X and Y are two distinct atoms of G, then $\nu(X \cap Y) \leq 2$.*

Lemma 2.7 *Let X and Y be two distinct atoms of graph G. If $\lambda_3(G) = 5$, then $\nu(X) \geq 5$ and $X \cap Y$ is empty.*

Proof Let H be an arbitrary connected subgraph of G of order 3 or 4. By computing $\partial(H)$, we find that H cannot be a fragment of G since $\lambda_3(G) = 5$ and $k = 4$. Therefore $\nu(X) = \nu(Y) \geq 5$. Suppose, to the contrary, the second assertion is not true. Define A, B, C and D as before. By Corollary 2.6, we have $\nu(X \cap Y) \leq 2$. Therefore

$$\nu(B) \geq 3 \quad \text{and} \quad \nu(C) \geq 3.$$

On the other hand, since $\partial(B) + \partial(C) \leq \partial(X) + \partial(Y) = 2\lambda_3(G)$, it follows that

$$\partial(B) \leq \lambda_3(G) \quad \text{or} \quad \partial(C) \leq \lambda_3(G).$$

Hence, B or C is a fragment by Lemma 2.4. But they are proper subgraphs of atom X or Y, which contradicts to the definition of atom. Lemma 2.7 follows.

Lemma 2.8 *Let X be an atom of G. If $\lambda_3(G) = 5$, then X is vertex transitive.*

Proof Let u and v be two arbitrary vertices of X. Since G is vertex transitive, there is an automorphism $\tau \in Aut(G)$ such that $\tau(u) = v$. Since $\tau(X)$ is also an atom of G with the property that $\tau(X) \cap X \neq \emptyset$, it follows that $\tau(X) = X$ by Lemma 2.7. Therefore, $\tau|_X$, the restriction of τ on X, is an automorphism of X. Hence, X is vertex transitive.

3. Restricted edge connectivity

Theorem 3.1 *A 4-regular connected vertex transitive graph G of girth three and order at least six either is maximal R_3-edge connected or has $\lambda_3(G) = 4$. Moreover, $\lambda_3(G) = 4$ if and only if*

G contains K_4.

Proof Suppose, to the contrary, that the first part is not true. Since $\xi_3(G)=6$, by Lemmas 2.2 and 2.3 we have $\lambda_3(G)=5$. Let X be an atom of G, then $\nu(X)\geq 5$. According to Lemma 2.8, X is an r-regular vertex transitive subgraph with $4>r>0$. Therefore

$$5=\lambda_3(G)=\partial(X)=(4-r)\nu(X)\geq(4-r)5\geq 5.$$

It follows that $r=3$ and $\nu(X)=5$. But this is impossible since any regular graph with odd order must have even valence.

For the second part, let X be a subgraph isomorphic to K_4. Since $|G|\geq 6$ and $k=4$, it follows that $G\setminus X$ contains neither isolated vertex nor isolated edges, which shows that $I(X)$ is a 3-restricted edge cut and that $\lambda_3(G)\leq\partial(X)=4$. But on the other hand, according to Lemmas 2.2 and 2.3 we have that $4=\lambda(G)\leq\lambda_3(G)$, the sufficiency thus follows.

Claim 1. If atoms of graph G have order at least five, then they are pairwise disjoint.

Let X and Y be two distinct atoms of graph G. If Claim 1 fails to be true, according to Corollary 2.6 we have $\nu(X\cap Y)\leq 2$, and so $\nu(X\cap Y^c)\geq 3$, $\nu(X^c\cap Y)\geq 3$. On the other hand, we have $\partial(X\cap Y^c)+\partial(X^c\cap Y)\leq\partial(X)+\partial(Y)=2\lambda_3(G)$, and so $\partial(X\cap Y^c)\leq\lambda_3(G)$ or $\partial(X^c\cap Y^c)\leq\lambda_3(G)$. By Lemma 2.4, $X\cap Y^c$ or $X^c\cap Y$ is a fragment of atom X or Y, Claim 1 follows from this contradiction.

Claim 2. If atom X has order at least five, then X is vertex-transitive.

Let u and v be two vertices of atom X and τ be an isomorphism of $Aut(G)$ such that $\tau(u)=v$. Clearly, $\tau(X)$ is also an atom of graph G such that $\tau(X)\cap X\neq\emptyset$, by Claim 1 we have $\tau(X)=X$. But $\tau|_X$ is an isomorphism of X, Claim 2 thus follows.

Now we continue to prove the necessity, let X be an atom of G. Since $\lambda_3(G)=4$ and $\xi_3(G)=6$, it follows that $\nu(X)\geq 4$. If $\nu(X)\geq 5$, by Claim 2, X is an r-regular vertex-transitive graph with $0<r<4$. Hence

$$4=\partial(X)=(4-r)\nu(X)>(4-r)4\geq 4.$$

This contradiction implies that $\nu(X)=4$. Since $\partial(X)=\lambda_3(G)=4$ and $\varepsilon(X)=6$, it follows that $X=K_4$, the necessity follows.

Theorem 3.2 *Cubic connected vertex-transitive graph of girth three and order at least six is maximal R_3-edge connected.*

Proof Let G be a graph depictured above, by Lemmas 2.2 and 2.3 we have

$$3=\lambda(G)\leq\lambda_2(G)\leq\lambda_3(G)\leq\xi_3(G)=3.$$

Theorem 3.2 follows.

Remark According to Theorem 3.2, each vertex of a 3-regular vertex-transitive graph of girth three is contained in a unique triangle, so, by contracting every triangle of this graph to a vertex we can get a simpler vertex-transitive graph. Conversely, every 3-regular vertex-transitive graph of girth three can be obtained by inserting a triangle in every vertex of a properly selected vertex-set of some vertex-transitive graph. To illustrate the existence such graphs and the truth of Theorem 3.2, we depicture two of these graphs, Figure 1 is a triangular prism, and Figure 2 is obtained by inserting a triangle in each of the three vertices of a triangle of the triangular prism.

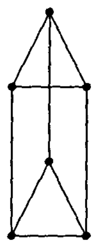

Figure 1

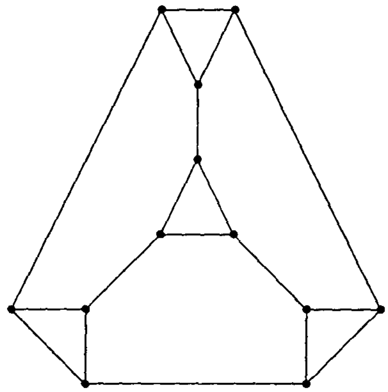

Figure 2

References:

[1] BAUER D, BOESCH F, SUFFEL C, et al. *Combinatorial optimization problem in the analysis and design of probabilistic networks* [J]. Networks, 1985, **15**: 257–271.

[2] ESFAHANIAN A H, HAKIMI S L. *On computing a conditional edge connectivity of a graph* [J]. Inform. Process. Lett, 1988, **27**: 195–199.

[3] LI Qiao-lian, LI Qiao. *Reliability analysis of circulant graphs* [J]. Networks, 1998, **28**: 61–65.

[4] OU JIANPING, ZHANG Fu-ji. *Sufficiency for the existence of R_m-restricted edge cut* [J]. J. Jishou University (Natural Science Edition), 2002, **1**: 21–24.

[5] MENG Ji-xian. *On a kind of restricted edge connectivity of graphs* [J]. Discrete Appl. Math., 2002, **117**(1-3): 183–193.

[6] WANG Ming, LI Qiao. *Conditional edge connectivity properties, reliability comparison and transitivity of graphs* [J]. Discrete Math., 2002, **258**: 205–214.

[7] OU Jian-ping, ZHANG FU-ji. *3-restricted edge connectivity of vertex-transitive graphs* [J]. ARS Combin., 2005, **70**: 1–11.

[8] BONDY J A, MURTY U S R. *Graph Theory with Its Application* [M]. London: Macmillan Press, 1976.

[9] MADER W. *Minmale n-fach kantenzusammenhängende graphen* [J]. Math. Ann., 1971, **191**: 21–28.

围长为 3 的点可迁图的 3 限制边连通度

欧见平 [1,2]， 张福基 [3]

(1. 五邑大学数理系，广东 江门 529020; 2. 漳州师院数学系，福建 漳州 363000;
3. 厦门大学数学系，福建 厦门 361005)

摘要: 设 G 是阶至少为 6 的 k 正则连通图. 如果 G 的围长等于 3, 那么它的 3 限制边连通度 $\lambda_3(G) \le 3k-6$. 当 G 是 3 或者 4 正则连通点可迁图时等号成立，除非 G 是 4 正则图并且 $\lambda_3(G)=4$. 进一步， $\lambda_3(G)=4$ 的充分必要条件是图 G 含有子图 K_4.

关键词: 点可迁图； 3 限制边连通度；限制断片.