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Some New Results on Double Domination in Graphs

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Abstract: Each vertex of a graph $G = (V, E)$ is said to dominate every vertex in its closed neighborhood. A set $S \subseteq V$ is a double dominating set for G if each vertex in V is dominated by at least two vertices in S. The smallest cardinality of a double dominating set is called the double dominating number $dd(G)$. In this paper, new relationships between $dd(G)$ and other domination parameters are explored and some results of [1] are extended. Furthermore, we give the Nordhaus-Gaddum-type results for double dominating number.

Key words: double domination number; claw-free graph; connected domination number.
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1. Introduction

Let $G = (V, E)$ be graph with $|V| = n$ and $|E| = m$. Each vertex of a graph is said to dominate every vertex in its closed neighborhood. A set $S \subseteq V$ is a dominating set if each vertex in V is dominated by some vertex of S. The domination number $\gamma(G)$ is the minimum cardinality of dominating set. Set S is a double dominating set for G if every vertex in V is dominated by at least two vertices in S. The minimum cardinality of a double dominating set is the double domination number, denoted by $dd(G)$. We refer to a minimum dominating set as γ-set and a minimum double dominating set as a dd-set.

A graph G is claw-free if it does not contain any $K_{1,3}$ as an induced subgraph. The degree, neighborhood and closed neighborhood of a vertex x in the graph G are denoted by $d(x), N(x)$ and $N[x] = N(x) \cup \{x\}$, respectively. For $X \subseteq V$, we write $N(X) = \cup_{x \in X} N(x)$ and $N[X] = N(X) \cup X$. Let $\delta(G)$ and $\Delta(G)$ denote the minimum degree and the maximum degree of the graph G. The graph induced by $X \subseteq V$ is denoted by $G[X]$. Let C_n and $K_{1,n-1}$ denote a cycle and star with n vertices, respectively. Let $\mathrm{diam}(G)$ denote the diameter of G, and let $d(u, v)$ denote the shortest distance between u and v.

Frank Harary and Teresa W. Haynes[1] initiate the study of double domination in graph. They present bounds and some exact values for $dd(G)$ and explore some relationships between $dd(G)$and other domination parameters. In this paper, new relationships between $dd(G)$ and other domination parameters are explored and some results of [1] are extended. Furthermore, we give the Nordhaus-Gaddum-type results for double dominating number.

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2. New relationships between double domination number and other domination parameters

We consider relationships between $dd(G)$ and the domination number $\gamma(G)$ as follows.

Lemma 1[1] *For any graph G with no isolated vertices, $\gamma(G) \le dd(G) - 1$ and this bound is sharp.*

But if G is claw-free, we can improve the bound.

Theorem 2 *Let G be a claw-free graph, then $\gamma(G) \le \frac{3dd(G)}{4}$ and the bound is sharp.*

Proof Let X be a dd-set of G, and let S' be a γ-set of $G[X]$. For $Q_1 = (V \setminus X) \setminus N(S')$, let Q_2 be a maximal independent set in $G[Q_1]$. Firstly, we claim that no two vertices in Q_2 have a common neighbor in X. Otherwise, assume that two vertices $q, q' \in Q_2$ have a common neighbor $x \in X$. For $y \in S' \cap N(x)$, the graph $G[q, q', x, y]$ is an induced claw in G which is a contradiction. Hence, no two vertices in Q_2 have a common neighbor in X. Since every vertex in Q_2 has at least 2 neighbors in X, we obtain $|Q_2| \le \frac{|X|-|S'|}{2}$. It is obvious that the set $S = S' \cup Q_2$ is a dominating set of G. So,

$$\gamma(G) \le |S| = |S'| + |Q_2| \le \frac{|X| - |S'|}{2} + |S'| = \frac{dd(G) + \gamma(G[X])}{2}.$$

Since the minimum degree of $G[X]$ is at least 1, it follows that

$$\gamma(G[X]) \le \frac{|X|}{2} = \frac{dd(G)}{2}.$$

Hence,

$$\gamma(G) \le \frac{dd(G) + \frac{dd(G)}{2}}{2} = \frac{3dd(G)}{4}.$$

The bound is sharp as can be seen by the graph G with vertex set $V(G) = \{y_i | 1 \le i \le 4\} \cup \{q_{ij} | 1 \le i < j \le 4\}$ and edge set $E(G) = \{y_1y_2, y_3y_4, q_{13}q_{23}, q_{23}q_{24}, q_{24}q_{14}, q_{14}q_{13}\} \cup \{q_{ij}y_i, q_{ij}y_j | 1 \le i < j \le 4\}$. Clearly, G is claw free and $dd(G) = 4$, $\gamma(G) = 3 = 4 \times \frac{3}{4}$.

S. T. Hedetniemi and Renu Lasker[2] define a connected dominating set S of G to mean S is a dominating set and $G[S]$ is connected. The minimum cardinality taken over all connected dominating sets is called the connected domination number of G, denoted by $\gamma_c(G)$.

Frank Harary and Teresa W. Haynes[1] show that no particular inequality holds between $dd(G)$ and $\gamma_c(G)$ by some examples.

Lemma 3[1] *$dd(C_n) = \lceil \frac{2n}{3} \rceil < n - 2 = \gamma_c(C_n)$ for $n \ge 9$; $dd(K_{1,m}) = m + 1 > 1 = \gamma_c(K_{1,m})$ for $m > 1$; $dd(C_6) = 4 = \gamma_c(C_6)$.*

However, we can obtain a general relationship between $dd(G)$ and $\gamma_c(G)$ by a method of [4].

Theorem 4 *For every connected graph $G = (V, E)$, $\gamma_c(G) \le 2dd(G) - 2$.*

Proof Let G be connected, and $X \subseteq V$ be a double dominating set of size $dd(G)$. Following [4], we construct a connected dominating set C from a double dominating set X by adding in every step at most 2 vertices to the double dominating set X such that at least two component of $G[X]$ form a connected component in $G[X']$, where X' is the union of X and the new vertices. This keeps $X := X'$ a double dominating set and we get a connected dominating set after at most number of components of X minus 1 steps (Note that two vertices connecting at least two components of $G[X]$ ever exist, since otherwise X would not be a double dominating set!)

Now since $G[X]$ has at most $\frac{|X|}{2}$ components, thus we get a connected dominating set $C \supseteq X$ by adding at most $2(\lfloor\frac{|X|}{2}\rfloor - 1) \leq |X| - 2$ vertices, consequently,

$$\gamma_c(G) \leq 2|X| - 2 = 2dd(G) - 2.$$

3. Nordhaus-Gaddum-type results for double dominating number

Nordhaus and Gaddum provided some best possible bounds on the sum of the chromatic numbers of a graph and its complement in [5]. A corresponding result for the domination number was presented by Jaeger and Payan[3]: If G is a graph of order $n \geq 2$, then $\gamma(G) + \gamma(\overline{G}) \leq n + 1$. An improved upper bound is due to Joseph and Arumugam: If G is a graph of order n such that neither G nor $\overline{G}$ has isolated vertices, then $\gamma(G) + \gamma(\overline{G}) \leq \frac{n+4}{2}$.

We now prove some best possible bounds on the sum of double domination numbers of a graph and its complement.

Lemma 5[1] *Let G be a graph with no isolated vertices. Then $2 \leq dd(G) \leq n$ and these bounds are sharp.*

Lemma 6[1] *Let G be a graph with $\delta(G) \geq 2$ and $n \neq 3, 5$. Then $dd(G) \leq \lfloor\frac{n}{2}\rfloor + \gamma(G) - 1$.*

Theorem 7 *If G is disconnected and with no isolated vertices, then $dd(\overline{G}) \leq 4$.*

Proof If G is disconnected, then let the components of G be $G_1, G_2, \cdots, G_w$. If $w \geq 3$ and $v_i \in V(G_i)$ for $i = 1, 2, 3$, then $\{v_1, v_2, v_3\}$ is a double dominating set of $\overline{G}$, so $dd(\overline{G}) \leq 3$. If $w = 2$, since G is a graph with no isolated vertices, $|V(G_i)| \geq 2$ for $i = 1, 2$. Let $v_{11}, v_{12} \in V(G_1)$ and $v_{21}, v_{22} \in V(G_2)$, then $\{v_{11}, v_{12}, v_{21}, v_{22}\}$ is a double dominating set of $\overline{G}$. So $dd(\overline{G}) \leq 4$.

Theorem 8 *Let G be a connected graph. If the diameter of G is at least 4, then $dd(\overline{G}) \leq 4$.*

Proof If G is connected, let u, v be two vertices of G such that the distance from u to v is diam(G), and assume the vertices sequence of the distance to be $u = v_0 v_1 \cdots v_d = v$. Hence $d \geq 4$ and $\{v_0, v_1, v_{d-1}, v_d\}$ is a double dominating set of $\overline{G}$. So $dd(\overline{G}) \leq 4$.

Theorem 9 *Let G be graph with no isolated vertices and $\Delta(G) < n - 1$. If the diameter of G or $\overline{G}$ is more than 2, Then $dd(G) + dd(\overline{G}) \leq n + 4$.*

Proof Since G is a graph with no isolated vertices and $\Delta(G) < n - 1$, $\overline{G}$ is a graph with no isolated vertices and $\Delta(\overline{G}) < n - 1$.

Case 1 G or $\overline{G}$ is disconnected. Without loss of generality, we assume that G is disconnected. Then by Lemma 5 and Theorem 7, $dd(G)+dd(\overline{G})\le n+4$.

Case 2 Both G and $\overline{G}$ are connected. If the $\mathrm{diam}(G)\ge 4$ or $\mathrm{diam}(\overline{G})\ge 4$, then by Lemma 5 and Theorem 8, $dd(G)+dd(\overline{G})\le n+4$.

Case 3 Both G and $\overline{G}$ are connected with $\mathrm{diam}(G)=3$ and $diam(\overline{G})=3$. Let u,v be two vertices of G such that $d(u,v)=3$, and let the family $W=\{V_0,V_1,V_2,V_3\}$ be the level decomposition of G with respect to u, where $V_i=\{x:d(u,x)=i\}$ for $i=0,1,2,3$. It is obvious that $\{u\}=V_0$ and $v\in V_3$.

Case 3.1 There exist two vertices u and v with $d(u,v)=3$ such that the level decomposition of G with respect to either u or v, say u, satisfy $|V_3|\ge 2$. If there exists a vertex $s\in V_3$ such that s is adjacent to only one vertex of V_2, say $t\in V_2$, and let $w\in V_3$ and $w\ne s$, then $\{u,s,t,w\}$ is a double dominating set of $\overline{G}$. So $dd(\overline{G})\le 4$. By Lemma 5, the theorem holds. If every vertex of V_3 is adjacent to at least two vertices of V_2, then let $V_{21}=\{x\in V_2$: there exists at least a vertex $y\in V_3$ such that y is not adjacent to $x\}$ and $V_{22}=\{x\in V_2:x$ is adjacent to every vertex of $V_3\}$. So $V_0\cup V_{22}\cup V_3$ is a double dominating set of $\overline{G}$ and $V_0\cup V_1\cup V_{21}\cup V_{22}$ is a double dominating set of G, so

$$dd(G)\le |V_0|+|V_1|+|V_{21}|+|V_{22}|$$

and

$$dd(\overline{G})\le |V_0|+|V_3|+|V_{22}|.$$

Hence

$$dd(G)+dd(\overline{G})\le |V_0|+|V_1|+|V_{21}|+|V_{22}|+|V_0|+|V_3|+|V_{22}|=|V(G)|+|V_{22}|+1.$$

If $|V_{22}|\le 3$, then the theorem holds. If $|V_{22}|>3$, then let $s,t\in V_{22}$ and $w\in V_3$. So $V_0\cup V_1\cup V_{21}\cup\{s,t,w\}$ is a double dominating set of G, $dd(G)\le |V_0|+|V_1|+|V_{21}|+3$. Hence,

$$dd(G)+dd(\overline{G})\le |V_0|+|V_1|+|V_{21}|+3+|V_0|+|V_3|+|V_{22}|=n+4.$$

Case 3.2 For arbitrary two vertices u and v with $d(u,v)=3$, the level decomposition of G with respect to arbitrary vertex u or v, say u, has $|V_3|=1$. That is to say $V_0=\{u\}$ and $V_3=\{v\}$.

Case 3.2.1 Either $d(u)=1$ or $d(v)=1$; Without loss of generality, we assume $d(u)=1$. Hence $|V_0|=|V_1|=|V_3|=1$. Let $V_{21}=\{x\in V_2:x$ is not adjacent to $v\}$, and let $V_{22}=V_2-V_{21}$. Since $V_{22}\ne\emptyset$, let $t\in V_{22}$. So, $V_0\cup V_1\cup V_{22}\cup V_3$ is a double dominating set of $\overline{G}$ and $V_0\cup V_1\cup V_{21}\cup V_3\cup\{t\}$ is a double dominating set of G, so

$$dd(G)\le |V_0|+|V_1|+|V_{21}|+|V_3|+1$$

and

$$dd(\overline{G})\le |V_0|+|V_1|+|V_{22}|+|V_3|.$$

Hence

$$dd(G) + dd(\overline{G}) \le |V_0| + |V_1| + |V_{21}| + |V_3| + 1 + |V_0| + |V_1| + |V_{22}| + |V_3| = n + 4.$$

Case 3.2.2 $d(u) \ge 2$ and $d(v) \ge 2$. That is to say $|V_1| \ge 2$ and $|V_2| \ge 2$. We have the following claims:

Claim 1 For each vertex $s \in V_1$, $d(s) \ge 2$. Otherwise assume $s \in V_1$ and $d(s) = 1$, then $d(v, s) \ge 4$, which is a contradiction.

Claim 2 For each vertex $s \in V_2$, $d(s) \ge 2$. Otherwise if there exists a vertex $s \in V_2$ and $d(s) = 1$, then $d(v, s) \ge 3$. So both u and s have distance from v at least 3, which is a contradiction.

Claim 3 For each vertex $s \in V_1$, there exists at least a vertex $t \in V_2$ which is adjacent to s. The proof is similar to that of Claim 2.

Let $V_{21} = \{x \in V_2 : x \text{ is not adjacent to } v\}$, $V_{11} = \{x \in V_1 : x \text{ is adjacent to at least a vertex } V_{21}\}$, $V_{22} = V_2 - V_{21}$, $V_{12} = V_1 - V_{11}$.

Claim 4 For each vertex $s \in V_{21}$, there exists at least a vertex $t \in V_{22}$ which is adjacent to s. The proof is similar to that of Claim 2.

Claim 5 For each vertex $s \in V(G)$, we can assume that $d_{\overline{G}}(s) \ge 2$. Clearly, $d_{\overline{G}}(u) \ge 2$ and $d_{\overline{G}}(v) \ge 2$. If there exists a vertex $s \in V_1$ such that $d_{\overline{G}}(v) = 1$, then there exist rwo verties $w, t \in V_2$ such that $d(s, w) \ge 3$ and $d(s, t) \ge 3$ in $\overline{G}$. Then replace G with $\overline{G}$, by Case 3.1, the theorem holds.

If $|V_{21}| \le 2$, then $\gamma(G) \le 4$. Clearly, $\gamma(\overline{G}) \le 2$. By Lemma 6, $dd(G) \le \lfloor \frac{n}{2} \rfloor + \gamma(G) - 1 \le \lfloor \frac{n}{2} \rfloor + 3$, $dd(\overline{G}) \le \lfloor \frac{n}{2} \rfloor + \gamma(\overline{G}) - 1 \le \lfloor \frac{n}{2} \rfloor + 1$. So

$$dd(G) + dd(\overline{G}) \le n + 4.$$

Hence, we can assume that $|V_{21}| \ge 3$. If there exists a vertex $s \in V_{11} \cup V_{22}$ such that s is adjacent to all vertices of $V_{12} \cup V_{21}$, then $\gamma(G) \le 3$. With a similar way as above, the theorem holds. If for arbitray vertex $s \in V_{11} \cup V_{22}$, there is a vertex $t \in V_{12} \cup V_{21}$ such that s is not adjacent to t, then by Claim 4, $V_0 \cup V_{11} \cup V_{22} \cup V_3$ is a double dominating set of G and $V_0 \cup V_{12} \cup V_{21} \cup V_3$ is a double dominating set of $\overline{G}$. So

$$dd(G) \le |V_0| + |V_{11}| + |V_{22}| + |V_3|$$

and

$$dd(\overline{G}) \le |V_0| + |V_{12}| + |V_{21}| + |V_3|.$$

Hence,

$$dd(G) + dd(\overline{G}) \le |V_0| + |V_{11}| + |V_{22}| + |V_3| + |V_0| + |V_{12}| + |V_{21}| + |V_3| = n + 2.$$

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图的双控制的一些新结果

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摘要: 图 $G=(V,E)$ 的每个顶点控制它的闭邻域的每个顶点. S 是一个顶点子集合，如果 G 的每一个顶点至少被 S 中的两个顶点控制，则称 S 是 G 的一个双控制集. 把双控制集的最小基数称为双控制数，记为 $dd(G)$. 本文探讨了双控制数和其它控制参数的一些新关系，推广了 [1] 的一些结果. 并且给出了双控制数的 Nordhaus-Gaddum 类型的结果.

关键词: 双控制数；无爪图；连通控制数.