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Ranks of Generalized Star Sign Patterns

GAO Yu-bin, SHAO Yan-ling
(Dept. of Math., North University of China, Taiyuan 030051, China)
(E-mail: ybgao@nuc.edu.cn)

Abstract: A sign pattern is a matrix whose entries are from the set $\{+,-,0\}$. A sign pattern is a generalized star sign pattern if it is combinatorial symmetric and its graph is a generalized star graph. The purpose of this paper is to obtain the bound of minimal rank of any generalized star sign pattern (possibly with nonzero diagonal entries).

Key words: sign pattern; generalized star sign pattern; minimal rank.
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1. Introduction

A sign pattern (matrix) A is a matrix whose entries are in the set $\{+,-,0\}$. Denote the set of all $n\times n$ sign patterns by Q_n. Associated with each sign pattern $A=(a_{ij})\in Q_n$ is a class of real matrices, called the sign pattern class of A, defined by

$$Q(A)=\{B=(b_{ij})\mid B \text{ is an } n\times n \text{ real matrix, and } \mathrm{sign} b_{ij}=a_{ij} \text{ for all } i \text{ and } j\}.$$

Let $A\in Q_n$. A has an identical zero determinant provided each of the $n!$ terms in the standard determinant expansion is 0. A is said to be sign nonsingular if each matrix $B\in Q(A)$ is nonsingular. It is well known that A is sign nonsingular if and only if $\det(A)=+$ or $\det(A)=-$, that is, in the standard expansion of $\det(A)$ into $n!$ terms, there is at least one nonzero term, and all the nonzero terms have the same sign.

Let $A=(a_{ij})\in Q_n$. If $a_{ij}\neq 0$ whenever $a_{ji}\neq 0$, then A is called combinatorially symmetric. For a combinatorially symmetric sign pattern $A\in Q_n$, by $G(A)$ we mean the undirected graph of A, with vertex set $\{1,\cdots,n\}$ and (i,j) is an edge if and only if $a_{ij}\neq 0$. We call $G(A)$ the graph of the sign pattern A.

For a sign pattern $A\in Q_n$, we define $\mathrm{mr}(A)$, the minimal rank of A by

$$\mathrm{mr}(A)=\min\{\ \mathrm{rank}(B)\mid B\in Q(A)\ \}.$$

Similarly, the maximal rank of A, $\mathrm{MR}(A)$, is

$$\mathrm{MR}(A)=\max\{\ \mathrm{rank}(B)\mid B\in Q(A)\ \}.$$

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The minimal rank and maximal rank of a sign pattern can be used to consider the inertia of the sign pattern[2].

In [2], it was proved that, for a combinatorially symmetric sign pattern A, $\mathrm{MR}(A)$ is the maximum number of nonzero entries of A with no two of the nonzero entries in the same row and column (it is the term rank of A as defined in [4]). But, in a general way, how to decide the minimal rank of a sign pattern is difficult. The purpose of this paper is to obtain the bound of minimal rank of a generalized star sign pattern (defined as follows).

Let $K_{1,n-1}$ be a star graph of order n. The graph G is obtained from $K_{1,n-1}$ by inserting some vertices on each edge. We call G a generalized star graph. A sign pattern $A \in Q_n$ is a generalized star sign pattern if A is combinatorially symmetric and $G(A)$ is a generalized star graph, possibly with loops.

For an $n \times n$ matrix or sign pattern A and nonempty subsets α and β of $\{1, 2, \cdots, n\}$, $A[\alpha, \beta]$ denotes the submatrix of A that lies in the rows of A indexed by α and the columns indexed by β. We also abbreviate $A[\alpha, \alpha]$ as $A[\alpha]$.

2. Main results

Theorem 2.1 *Let $A \in Q_n$ with $\mathrm{mr}(A) = m$ and $\mathrm{MR}(A) = M$. Then there exists a real matrix $B \in Q(A)$ such that $\mathrm{rank}(B) = k$ for any $m \le k \le M$.*

Proof Clearly, there exist two matrices $B_1, B_2 \in Q(A)$ such that $\mathrm{rank}(B_1) = m$ and $\mathrm{rank}(B_2) = M$. Denote the row vectors of B_1 as $\alpha_1, \alpha_2, \ldots, \alpha_n$, and the row vectors of B_2 as $\beta_1, \beta_2, \ldots, \beta_n$. For $2 \le k \le n$, let C_k be a real matrix with row vectors $\beta_1, \ldots, \beta_{k-1}, \alpha_k, \ldots, \alpha_n$, and $C_1 = B_1$ and $C_{n+1} = B_2$. Thus $C_i \in Q(A)$ for $i = 1, 2, \ldots, n+1$. It is not difficult to verify that

$$| \mathrm{rank}(C_i) - \mathrm{rank}(C_{i+1}) | \le 1, \quad \text{for } i = 1, 2, \ldots, n.$$

Thus the theorem follows. □

Lemma 2.2 *Let B be a real matrix of order n as follows*

$$B = \begin{bmatrix} c_1 & a_1 & & & \\ b_1 & c_2 & a_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & a_{n-1} \\ & & & b_{n-1} & c_n \end{bmatrix},$$

where $a_i \neq 0$ and $b_i \neq 0$ for $i = 1, 2, \ldots, n-1$. If $\mathrm{rank}(B) = n-1$, then the first row (column) vector of B can be linearly represented by the other row (column) vectors of B.

Proof Denote the row vectors of B by $\alpha_1, \alpha_2, \ldots, \alpha_n$. Since $\mathrm{rank}(B) = n-1$, the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly dependent, that is, there exist n numbers $k_1, k_2, \ldots, k_n$ which are not all zeros such that $k_1\alpha_1 + k_2\alpha_2 + \ldots + k_n\alpha_n = 0$. If $k_1 = 0$, then, by $b_i \neq 0$ for $i = 1, 2, \ldots, n-1$, we have $k_2 = k_3 = \ldots = k_n = 0$, a contradiction. Thus $k_1 \neq 0$, and α_1 can be linearly represented by

$\alpha_2, \ldots, \alpha_n$. Similarly, we can prove that the first column vector of B can be linearly represented by the other column vectors of B. The theorems follow. □

Theorem 2.3 *Let $A \in Q_n$ whose graph $G(A)$ is a path, possibly with loops. Then*

$$n-1 \le \mathrm{mr}(A) \le \mathrm{MR}(A) \le n.$$

Proof Without loss of generality, we may assume that A is a tridiagonal sign pattern. For any $B \in Q(A)$, it is clear that

$$\det B[\{1,2,\ldots,n-1\},\{2,3,\ldots,n\}] \ne 0.$$

Then $n-1 \le \mathrm{rank}(B) \le n$, so the theorem holds. □

From Theorem 2.3, the following theorem is clear, and we may omit the proof.

Theorem 2.4 *Let $A \in Q_n$ whose graph $G(A)$ is a path, possibly with loops. Then $\mathrm{mr}(A) = n-1$ if and only if A is not sign nonsingular, that is, one of the following holds.*

(1) A has an identically zero determinant.

(2) In the standard expansion of $\det(A)$ into $n!$ terms, there are at least two nonzero terms, one is $+$ and the other one is $-$.

Theorem 2.5 *Let $A \in Q_n$ whose graph $G(A)$ is a generalized star graph in Fig. 1, possibly with loops, where $n = n_1 + n_2 + \ldots + n_k + 1$, $k \ge 3$ and $n_i \ge 1$ for $i = 1,2,\ldots,k$. Then*

$$n-k+1 \le \mathrm{mr}(A) \le \mathrm{MR}(A) \le n.$$

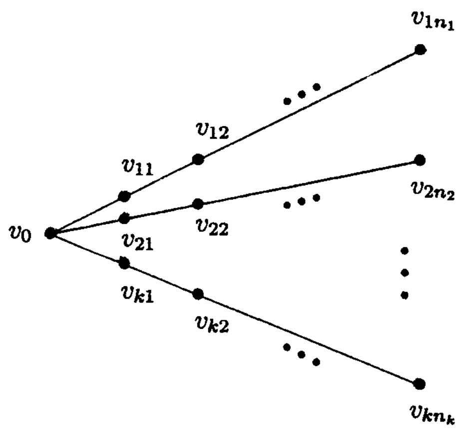

Fig. 1 Graph $G(A)$

Proof It is clear that $\mathrm{mr}(A) \le \mathrm{MR}(A) \le n$. We need only prove that $\mathrm{mr}(A) \ge n-k+1$.

Without loss of generality, we order vertices of $G(A)$ as v_0, $v_{11}, v_{12}, \ldots, v_{1n_1}$, $v_{21}, v_{22}, \ldots,$

$v_{2n_2}, \ldots, v_{k1}, v_{k2}, \ldots, v_{kn_k}$. Then A has the following form

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} & \cdots & A_{0k} \\ A_{10} & A_{11} & 0 & \cdots & 0 \\ A_{20} & 0 & A_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ A_{k0} & 0 & \cdots & 0 & A_{kk} \end{bmatrix}, \tag{2.1}$$

where A_{ii} is a tridiagonal sign pattern of order n_i for $i = 0, 1, 2, \ldots, k$ with $n_0 = 1$, and A_{0j} and A_{j0} are $1 \times n_j$ and $n_j \times 1$ sign patterns, respectively, with the first entry is nonzero and other entries are zero for $j = 1, 2, \ldots, k$.

Let $\overline{A} = A[\{2, 3, \ldots, n\}]$. Then $\overline{A}$ is a block diagonal sign pattern, and each diagonal block of $\overline{A}$ is a tridiagonal sign pattern. Clearly, it follows that

$$\mathrm{mr}(A) \geq \mathrm{mr}(\overline{A}) = \sum_{i=1}^{k} \mathrm{mr}(A_{ii}), \tag{2.2}$$

and

$$n_i - 1 \leq \mathrm{mr}(A_{ii}) \leq n_i, \quad i = 1, 2, \ldots, k. \tag{2.3}$$

Now we consider the following two cases.

Case 1. $\mathrm{mr}(A_{ii}) = n_i$ for $i = 1, 2, \ldots, k$.

By (2.2), it is clear that $\mathrm{mr}(A) \geq \sum_{i=1}^{k} \mathrm{mr}(A_{ii}) = \sum_{i=1}^{k} n_i = n - 1 > n - k + 1$. The theorem holds.

Case 2. There exists $1 \leq s \leq k$ such that $\mathrm{mr}(A_{ss}) = n_s - 1$.

For any real matrix $B \in Q(A)$, by (2.1), B has the following form

$$B = \begin{bmatrix} B_{00} & B_{01} & B_{02} & \cdots & B_{0k} \\ B_{10} & B_{11} & 0 & \cdots & 0 \\ B_{20} & 0 & B_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ B_{k0} & 0 & \cdots & 0 & B_{kk} \end{bmatrix}, \tag{2.4}$$

where $B_{ij} \in Q(A_{ij})$, and $\mathrm{rank}(B_{ss}) = n_s - 1$. By Lemma 2.2, the first row vector of B_{ss} can be linearly represented by the other row vectors of B_{ss}, and the first column vector of B_{ss} can be linearly represented by the other column vectors of B_{ss}. Thus there exist nonsingular matrices P and Q of order n such that

$$PBQ = C = \begin{bmatrix} 0 & & & C_{0s} & & \\ & C_{11} & & & & \\ & & \ddots & & & \\ C_{s0} & & & C_{ss} & & \\ & & & & \ddots & \\ & & & & & C_{kk} \end{bmatrix}, \tag{2.5}$$

where $C_{s0} = B_{s0}$, $C_{0s} = B_{0s}$, $C_{ii} = B_{ii}$ for $1 \le i \le k$ and $i \ne s$, C_{ss} is a matrix obtained from B_{ss} by replacing the first row and first column of B_{ss} by zero. In this case, we have that $\mathrm{rank}(B) = \mathrm{rank}(C)$, and $\mathrm{rank}(B_{ii}) = \mathrm{rank}(C_{ii})$ for $1 \le i \le k$. Then

$$\mathrm{rank}(B) = \sum_{i=1}^{k} \mathrm{rank}(C_{ii}) + 2 \ge \sum_{i=1}^{k} (n_i - 1) + 2 = n - k + 1.$$

The theorem follows. □

Theorem 2.6 *Let $A \in Q_n$ be a generalized star sign pattern having the following form*

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} & \cdots & A_{0k} \\ A_{10} & A_{11} & 0 & \cdots & 0 \\ A_{20} & 0 & A_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ A_{k0} & 0 & \cdots & 0 & A_{kk} \end{bmatrix}, \quad (2.1)$$

where A_{ii} is a tridiagonal sign pattern of order n_i ($n_i \ge 1$) for $i = 0, 1, 2, \ldots, k$ with $n_0 = 1$, $n = n_1 + n_2 + \ldots + n_k + 1$ and $k \ge 3$, and A_{0j} and A_{j0} are $1 \times n_j$ and $n_j \times 1$ sign patterns, respectively, with the first entry is nonzero and other entries are zero for $j = 1, 2, \ldots, k$. Then $\mathrm{mr}(A) = n - k + 1$ if and only if $\mathrm{mr}(A_{ii}) = n_i - 1$ for $i = 1, 2, \ldots, k$.

Proof It is clear from the proof of Theorem 2.4. □

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广义星符号模式的秩

高玉斌， 邵燕灵
(中北大学数学系，山西 太原 030051)

摘要: 一个符号模式是一个元素取自于集合 $\{+,-,0\}$ 的矩阵. 如果符号模式 A 是组合对称的，且它的图是一个广义星图，则称 A 是广义星符号模式. 对于任意的广义星符号模式 (可能有非零对角元), 本文给出其最小秩的界.

关键词: 符号模式；广义星符号模式；最小秩.