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# An Interpolation Theorem for Near-Triangulations

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**Abstract**: A near-triangular embedding is an embedded graph into some surface whose all but one facial walks are 3-gons. In this paper we show that if a graph $G$ is a triangulation of an orientable surface $S_h$, then $G$ has a near-triangular embedding into $S_k$ for $k = h, h+1, \cdots, \lfloor \frac{\beta(G)}{2} \rfloor$, where $\beta(G)$ is the Betti number of $G$.

**Key words**: triangulation; embedding; orientable surface.
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## 1. Introduction

Throughout this paper we study simple connected graphs. Terms and notations are standard and the readers may refer[1] for used concepts but undefined.

A surface is a compact 2-manifold without boundary. It is well-known that all the surfaces are classified into two types: orientable surface which is the sphere with $h$ handles, denoted by $S_h$, and nonorientable surface $N_h$, the sphere with $h$ crosscaps. An embedding of a graph $G$ into a surface $\Sigma$ is a drawing of $G$ in $\Sigma$ such that no edge-crossing is permitted and each component of $\Sigma - G$ is an open disc called face. The bounary of a face is called a facial-walk. If the length of a facial walk is $k$, then the face is a $k$-gon. A cycle $C$ of an embedded graph $G$ in a surface $\Sigma$ is called essential if no component of $\Sigma - C$ is homoeomorphic to a disc; otherwise it is tirvial.

A near-triangulation of a surface is an embedded graph $G$ in the host surface such that all but one faces are 3-gons and the embedding is called near-triangular-embedding or $G$ is a near-triangulated graph. If each face is a 3-gon, then $G$ is a triangulation.

A remarkable result on orientable embeddings is the following Duke's result[2]:

**Theorem A** *If a graph $G$ may be embedded into $S_m$ and $S_n (m \leq n)$, then $G$ may be embedded into $S_k$ for $m \leq k \leq n$.*

Although Duke's result shows the existence of embeddings with interpolation property, no more things for desribing the interpolated embeddings is provided in detail. What can we say more about such embeddings? In this paper we investigate the structures of the possible intermediate embeddings other than the minimum genus and maximum genus embeddings and

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establish an interpolation result for near-triangular-embeddings, i.e., the following result:

**Theorem B** *Let $G$ be a triangulation of some orientable surface $S_h$. Then $G$ may be embedded into $S_k$ for $h \le k \le \lfloor \frac{\beta(G)}{2} \rfloor$ such that all but one faces are 3-gons, where $\beta(G)$ is the Betti number of $G$.*

It is well-known that the maximum genus of a graph $G$ may not exceed $\lfloor \frac{\beta(G)}{2} \rfloor$. If the maximum genus attains $\lfloor \frac{\beta(G)}{2} \rfloor$, then we say that $G$ is up-embeddable. It is clear that if a graph is up-embbedable, then the corresponding embedding will have no more than two faces. One may readily see from the calculation of $\beta(G)$ that if a triangulation is up-embeddable, then the maximum genus embbedding should have exactly two faces. A basic result for up-embelibility of graphs is due to Nebeskỳ[3] which says that a locally connected graph is up-embbedable. Also it fails to provide more information for the maximum genus embeddings. As a direct consequence of Theorem B we have the following result:

**Corrollary** *If a graph $G$ is a triangulation, then $G$ may be embedded into $S_{\lfloor \frac{\beta(G)}{2} \rfloor}$ such that a face is a 3-gon.*

## 2. Planar triangulations

In this section we shall concentrate on the planar case of Theorem B which may help us to prove the result in general situation. Before our proofs we have to give some basic results on triangulations.

**Lemma 1**[4] *Any triangulation is a 3-connected graph.*

By a contractible edge $e$ in a 3-connected graph $G$ we mean that after shrunk the edge, the resulting graph $G \bullet e$ is also 3-connected. There has been much work down in studies of contractible edges but the most influencial is the following result due to C.Thomassen[5].

**Lemma 2** *Let $G$ be a 3-connected graph. Then there exists a contractible edge in $G$.*

Further, the result coming up next says that most 3-connected graphs will have several independently contractible edges (i.e., edges without vertex in common).

**Lemma 3**[6] *Let $G$ be a 3-connected graph with no less than five vertices and $e$ be a contractible edge. Then $G$ contains another contractible edge which has no vertex in common with $e$.*

As a direct consequence of Lemmas 1-3 we have the following result.

**Lemma 4** *Let $G$ be a triangulation with more than four vertices. Then for any 3-cycle $C_3$ in $G$, there exists a contractible edge in $G$ which has no edge in common with $C_3$.*

The following result provides some conditions for an edge to be contractible.

**Lemma 5** *Let $G$ be a triangulation. If an edge $e$ is contained in exactly two 3-cycles (i.e., 3-gons), then $e$ is a contractible edge in $G$. Further, an edge of $G$ is contractible iff it is on the common boundary of exact two 3-cycles provided $G$ is a planar triangulation.*

Now we start to investigate a planar triangulation.

Let $G$ be a planar triangulation. It is easy to see that $G$ may be near-triangularly embedded into $S_k$ for $0 \leq k \leq \lfloor \frac{\beta(G)}{2} \rfloor$ when the number of edges is smaller (i.e., Theorem B stands for smaller planar triangulations). Suppose that Theorem B holds for the planar triangulations with less than $q$ edges. We now consider a planar triangulation $G$ with $q$ edges. By Lemmas 1-5, $G$ contains a contractible edge $e = (x, y)$ lying on the boundary of exact two 3-gons $C_3' = (x, y, z)$ and $C_3'' = (x, y, u)$. Let $P = x, v_{xy}, u$ be the path in $G \bullet e$ from $z$ to $u$ via the vertex $v_{xy}$, the new vertex pertaining to $e$. Then $G \bullet e$ is also a planar triangulation with fewer edges than $q$. By induction hypothesis $G \bullet e$ may be near-triangularly embedded into $S_k$ for $0 \leq k \leq \frac{\beta(G \bullet e)}{2}$. Let $\Pi_k'$ be a such embedding of $G \bullet e$ into $S_k$. Then we may split the vertex $v_{xy}$ of $G \bullet e$ and recover the two 3-gons $C_3'$ and $C_3''$ in $G$ and get a near-triangular embedding $\pi_k$ of $G$ in $S_k$. In particular, $\pi'_{\lfloor \frac{\beta(G \bullet e)}{2} \rfloor}$ will induce a near-triangular embedding $\pi_{\lfloor \frac{\beta(G \bullet e)}{2} \rfloor}$ of $G$ into $S_{\lfloor \frac{\beta(G \bullet e)}{2} \rfloor}$ which has exact four faces among which three are 3-gons and further, the edge $e$ is on the common boundary of two 3-gons $C_3'$ and $C_3''$ in $\pi_{\lfloor \frac{\beta(G \bullet e)}{2} \rfloor}$. Since there are three distinct faces around one end of $e$, say $x$, we may carry on a *handle-increasing operation* (as depicted in Fig.1) which will merge the three faces into a bigger one and embedded $G$ into $S_{\lfloor \frac{\beta(G)}{2} \rfloor}$ such that there exists a 3-gon. By induction principle, Theorem B is valid for planar triangulations.

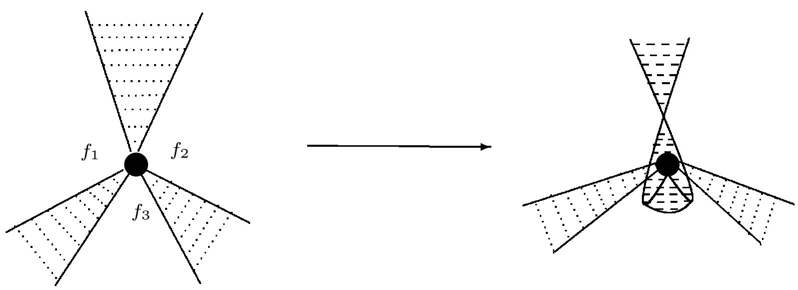


Fig.1 (A handle-increasing operation.)

## 3. Toroidal graphs

In this section we shall prove Theorem B in the case of toroidal triangulations which will provide much more information for our proof of Theorem B in general case.

Also we will proceed our proof by applying induction for the number of edges of triangulations considered. It is clear that Theorem B stands for smaller triangulations. Suppose that it holds for the toroidal triangulations with fewer edges than $q$. Now let us consider a toroidal triangulation $G$ with $q$ edges. We may further suppose that each edge of $G$ is contained in at least three 3-cycles since otherwise $G$ will have a contractible edge $e$ lying on the common boundary of exact two 3-gons which will imply near-triangular embeddings of $G$ into $S_k$ for $1 \leq k \leq \lfloor \frac{\beta(G)}{2} \rfloor$ provided we repeat the shrinking-splitting operations (togeter with a handle-increasing operation if possible) as we have reasoned in §2.

Let $e$ be a contractible edge of $G$. Then the next procedure will be handled in several cases according to the situations of essential 3-cycles passing through $e$.

**Case 1** There is an essential 3-cycle containing $e$ in $G$ such that no other 3-cycle containing $e$ is homotopic to it.

Here we mean a pair of essential cycles, say $C, C'$, of an embbeded graph in an orientable surface to be homotopic if and only if cutting along $\{C, C'\}$ results in a graph which has a component $\sigma$ which is planar graph and contains precisely one copy of $C$ and one copy of $C'$.

Let $C_3 = (x, y, z)$ be the essential 3-cycle of $G$ containing $e$ as assumed. Although there may be another essential 3-cycle passing through $e$ in $G$, cutting $S_1$ along $C_3$ will destroy all types of such cycles since all other essential 3-cycles passing through $e$ share the property that they contain, respectively, a path from $x$ to $y$ which starts from an edge in the *right graph* $G_r(C_3)$ and ends at another edge in the left graph $G_l(C_3)$. Here we use the concepts of the right graph and the left graph as defined in [4] or [6]. Then cutting $S_1$ along $C_3$ will result in a planar triangulation $G_1$ in which the two copies $e'$ and $e''$ of $e$ are, respectively, contractible since they are, respectively, contained in exact two 3-cycles in $G_1$. By the result in §2, $G_1 \bullet e' \bullet e''$ may be near-triangularly embedded into $S_k$ for $1 \leq k \leq \lfloor \frac{\beta(G_1 \bullet e' \bullet e'')}{2} \rfloor$. Let $\pi'_k$ be such a embedding. Now we may split the two new vertices $v_{e'}$ and $v_{e''}$ (pertaining to $e'$ and $e''$, respectively) of $G_1 \bullet e' \bullet e''$ and then identify the two copies of $C_3$ of $G_1$ and finally get a near-triangular embedding of $G$ into $S_{k+1}$. In particular, for the embedding induced (this way) from the largest genus near-triangular embedding of $G_1$, we may carry on a handle- increasing operation at the three faces around the one end of $e$ and get a maximum genus embedding (as desired) of $G$ which has only two faces among which one is a 3-gon.

**Remark** In our proof above we show that a near-triangular embedding of $G_1 \bullet e' \bullet e''$ into $S_k$ may determine a near-triangular embedding of $G$ into $S_{k+1}$. In particular, a maximum genus embedding of $G_1 \bullet e' \bullet e''$ induces a near-triangular embedding of $G$ into $S_{\lfloor \frac{\beta(G \bullet e' \bullet e'')}{2} \rfloor + 1}$ which has four faces among which there are three 3-gons. In order to embedd $G$ into a higher surface, $S_{\lfloor \frac{\beta(G \bullet e' \bullet e'')}{2} \rfloor + 2}$, one have to carry through an handle-increasing operation to merge the four faces around one end of $e$ into a bigger one. Those two kinds of procedures are the key steps in our proofs. Further, a triangulation on higher orientable surface may need several more such handle-increasing operations.

**Case 2** There is a pair of essential homotopic 3-cycles passing through $e$ in $G$.

Let $C_3^1, C_3^2, \cdots, C_3^m$ be a maximal set of essential homotopic 3-cycles containing $e$ in $G$. Then $m \geq 2$.

**Subcase 2.1** $m = 2$.

Cutting $S_1$ along $C_3^1, C_3^2$ will separate $S_1$ (hence consequently $G$) into two pieces $G_0$ and $G_1$ such that $G_0$ is the triangulation determined by the section from $C_3^1$ to $C_3^2$ as depicted in Fig.2.

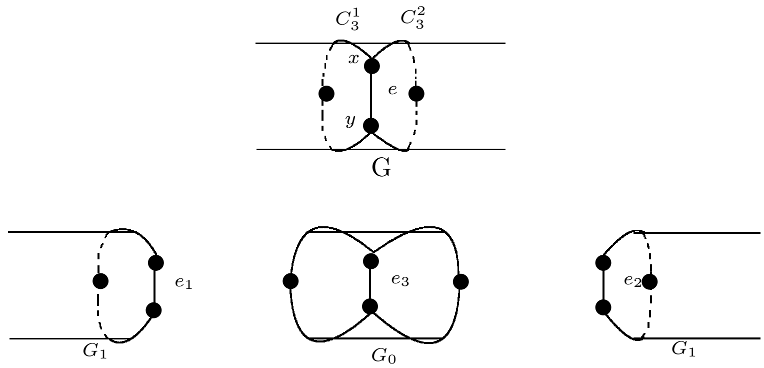


Fig.2 ( A triangulation $G_0$ induced by $G_r(C_3^1) \cap G_l(C_3^2)) \cup C_3^1 \cup C_3^2$)

It is clear that $G_0$ is a planar triangulation in which the copy of $e$ is also contractible since otherwise there are at least three essential homotopic 3-cycle passing through $e$ in $G$. Further, we may conclude that $G_0$ is a $K_4$, the complete graph with four vertices. If not so, the number of vertices of $G_0$ will be at least four. As a consequence of Lemma 3, $G_0$ will have another contractible edge which is also contractible and is on the common boundary of exact two 3-cycles in $G$, contrary to our assumption that $G$ has no such edges. Now let us see $G_1$. It is also a planar triangulation such that the two copies of $e$, say $e_1$ and $e_2$, in $G_1$ are, respectively, on the boundary of exact two 3-cycles. By applying the result in §2 and the shrinking-splitting procedure as used in case 1 we may conclude that $G_1 \bullet e_1 \bullet e_2$ may be near-triangularly embedded into $S_k$ for $0 \leq k \leq \lfloor \frac{\beta(G_1 \bullet e_1 \bullet e_2)}{2} \rfloor$ and for each of such embedding $\pi'_k$, splitting the two vertices (pertaining to $e'$ and $e''$, respectively) and then identifying the two copies of $C_3^i$ of $K_4$ and $G_1$ will naturally determine a near-triangular embedding $\pi_k$ of $G$ into $S_{k+1}$. In particular, by introducing the handle-inceasing operation twice at the faces around one end of $e$ (as we did in case 1 ) we may see that the largest genus embedding of $G_1 \bullet e_1 \bullet e_2$ will induce a near-triangular embedding of $G$ into the surface $S_{\lfloor \frac{\beta(G_1 \bullet e_1 \bullet e_2)}{2} \rfloor + 2}$ and $S_{\lfloor \frac{\beta(G_1 \bullet e_1 \bullet e_2)}{2} \rfloor + 3}$.

**Subcase 2.2** $m \geq 3$.

As we do in subcase 2.1, cutting $S_1$ along $C_3^1$ and $C_3^m$ will separate $S_1$ into two pices: one is a planar triangulation $G_1$ which is bounded by the section of $S_1$ between $C_3^1$ and $C_3^m$ while the other is a planar triangulation $G_2$. It is clear that the number of vertices of $G_1$ is at least 5. By Lemma 4, $G_1$ has at least two edge-disjoint contractible edges. If there is a contractible edge which is not on the copies of $C_3^1$ and $C_3^m$ in $G_1$, then it is on exact two 3-cycles of $G_1$ and so is for $G$. This is contrary to our assumption that any edge of $G$ is contained in three 3-cycles. So, we may suppose that the two egde-disjiont contractible edges of $G_1$ are contained in the two copies of $C_3^1$ and $C_3^m$, respectively. Further, let $C'_3$ and $C''_3$ be the two copies of $C_3^m$ in $G_1$ and $G_2$, respectively. Suppose further that $e'$ is the contractible edge of $G_1$ which is on $C'_3$. It is clear that the copy, say $e''$, of $e$ in $G_2$ and $e'$ are, respectively, on the common boundary of exact two 3-cycles in $G_1$ and $G_2$. Now let us consider the planar triangulation $G_0$ obtained by cutting $S_1$ along $C_3^m$. We conclude that both $e'$ and $e''$ are on the boundary of exact two 3-cycles in $G_0$.

By repeating the shrinking-splitting procedure and then identifying the copies of $C_3^m$ (together with a handle-increasing operation in the case of upper-embbeding) as used in case 1 or subcase 2.1 for $G_0 \bullet e' \bullet e''$ one may see that $G$ may be embbeded into $S_k$ for $1 \leq k \leq \lfloor \frac{\beta(G\bullet e'\bullet e'')}{2} \rfloor + 1$ such that all but one faces are triangular.

Based on the above statement, the proof of the case of toroidal triangulation of Theorem B is completed. $\square$

**Remark** (1) One may notice that we do not verify our toroidal case in the initial step of applying induction for the number of edges. In fact this has been concluded in our proving procedure since the smallest triangulations on the torus must share the property that each edge is contained in at least three 3-cycles which, through surgical operations on handles, may be reduced to the lower surfaces (here is the planar triangulations handled in the §2); (2) A reader with care may see that the shrinking-splitting operations are used for distinct pair of edges in the above two subcases. In fact, it is used for the two copies of the contractible edge $e$ in subcase 2.1 while in subcase 2.2 the contractible edges are, respectively, the copies of $e$ and an edge on $C_3^m$ which is incident to $e$ in $G$.

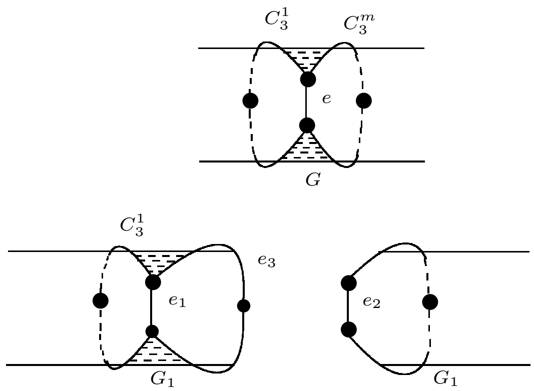


Fig.3 (Cutting along the right most cycle $C_3^m$ will result in a pair of copies $e_1, e_2$ of $e$ together with an edge $e_3$ which is on the a copy of $C_3^m$ such that $e_2, e_3$ are, respectively, in exact two 3-cycles in $G_1$ and treated as the candidates of contractible edges in handle-increasing operations.)

## 4. Higher orientable surfaces

In this section we shall concentrate on the triangulations on the orientable surfaces with more than one handles. Suppose that Theorem B holds for triangulations on the orientable surfaces with fewer than $g(\geq 2)$ handles. Now we consider the triangulations on $S_g$. Also we will proceed the proof by applying induction for the number of edges of a given triangulation. It is clear that Theorem B stands for smaller triangulations on $S_g$. Suppose that it is valid for the triangulations with fewer than $q$ edges. Now we consider a triangulation $G$ of $S_g$ with $q$ edges. We may suppose further that $G$ has no edges contained in exact two 3-cycles since otherwise we may apply induction hypothesis for the resulting graph $G \bullet e$ where $e$ is such a contractible edge.

Let $e$ be a contractible edge in $G$ by Lemmas 1 and 2. Then there are at lest three 3-cycles passing through $e$. Let $C_3, C_3'$ be a pair of essential 3-cycles passing through $e$. Then $E(C_3) \cap E(C_3') = \{e\}$. We say that $C_3'$ is *on the same side of* $C_3$ if the other two edges of $C_3'$ is on the same side of $C_3$ (under the local permutations of the edges incident to the two ends of $e$) or as defined in [4,6], the two ends of $C_3'$ are both contained in $G_l(C_3)$ or $G_r(C_3)$. In this case we call that the two 3-cycles are parallel. One should notice that a pair of parallel 3-cycles like this needn't to be homotopic since cutting along such cycles may not separate the host surface (as dipected in the right hand side of Fig.4).


Fig.4 (two types of parallel 3-cycles passing through a contractible edge e.)

Let $\mathcal{C} = \{C_3^1, C_3^2, \cdots, C_3^m\}$ be the set of essential 3-cycles passing through $e$ such that they are parallel to each other. Also we may suppose that $C_3^i$ ($C_3^j$) is on the left (right) side of $C_3^j$ provided $i$ is smaller (greater) than $j$. Then we conclude the following result.

**Lemma 6** *A pair of parallel essential cycles in $\mathcal{C}$ are homotopic iff cutting along those two cycles will separate the host surface such that one component is a planar graph containing two copies of the* 3*-cycles which are the boundaries of two faces, respectively.*

If there are some pair of 3-cycles in $\mathcal{C}$ which are also homotopic, then we may find a maximal group of pairwisely homotopic cycles in $\mathcal{C}$ which are contained in the section bounded by $C_3^i$ and $C_3^j$ (called the *boundary* of the section) such that no other cycles in $\mathcal{C}$ will be parallel to those lying between $C_3^i$ and $C_3^j$ for some integers $i$ and $j$. Thus we may find a cllection of maximal groups of homotopic cycles in $\mathcal{C}$ (each cycle in $\mathcal{C}$ may have only one element in $\mathcal{C}$) which may classified into three possible types. One type of the maximal group consists a single element; each of the second type have exact two elements which bound a $K_4$ (as we have shown in subcase 2.1 in §2); every group of the third type, as we have analysed in subcase 2.2 of the §2, contains at least three elements in $\mathcal{C}$ which determine a planar triangulation with at least five vertices whose possible two edge-disjoint contractible edges are, respectively, contained in the two copies of the right most 3-cycle in $G$.

Now we will introduce several surgical operations.

For a group $\mathcal{C}_1$ of the first type, we cut along the only element in $\mathcal{C}_1$ which will result in a pair of copies of the edge $e$ which are, respectively, contained in exact two 3-cycles in the resulting graph and hence are the candidates of contractible edges of the shrinking-splitting proceedures and the handle-increasing operations.

Let $\mathcal{C}_2$ be a group of the second type. Then (as we have reseaoned in subcase 2.1) cutting along the two elements in $\mathcal{C}_2$ will increase a component $K_4$ and increase three copies of $e$. The two copies of $e$ contained in the same component are treated as the candidates of contractible edges in our shrinking-splittings operations and the handle-increasing operations..

Let $\mathcal{C}_3$ be a group of the third type whose elements are listed as $C_3^{i_1}$, $C_3^{i_2}, \cdots, C_3^{i_j}$ such that ${C_3}^{i_s}$ is on the left side of ${C_3}^{i_t}$ for $i_s < i_t$. One may readily see that this situation is in fact the case we have handled in subcase 2.2 in §3. Then we cut along $C_3^{i_j}$ and get two copies of $C_3^{i_j}$ such that one copy contains a copy of $e$ which is a contained in exact two 3-cycles while the other copy, as we have analysed in subcase 2.2, contains a contractible edge (which is not a copy of $e$) which is also contained in exact two 3-cycles. Those two egdes are treated as the candidates of the contractible edges for the shrinking-splitting and the handle-increasing procedures.

Let $\mathcal{E}$ be the set of all the contractible edges obtained in the above three kinds of cuttings. Then we consider the resulting component $G_1$ which is the possible non-$K_4$ triangulation of an orientable surface $S_h$. We may apply the induction hypothesis for $G_1 \bullet \mathcal{E}$ and embed it into $S_k$ for $h \le k \le \lfloor \frac{\beta(G_1 \bullet \mathcal{E})}{2} \rfloor$ such that all but one faces are triangular, where $G_1 \bullet \mathcal{E}$ is the graph obtained by shrinking all the edges of $\mathcal{E}$. For each near-triangular embedding $\pi'_k$ of $G_1 \bullet \mathcal{E}$, we split the new vertices (pertaining to the edges in $\mathcal{E}$) and then identify all the copies of the cycles in $\mathcal{C} = \{C_3^1, C_3^2, \cdots, C_3^m\}$ which may determine an embedding $\pi_k$ of $G$ in $S_{\lfloor \frac{\beta(G_1 \bullet \mathcal{E})}{2} + l_1 + l_2 + l_3 \rfloor}$ such that all but one faces are 3-gons where $l_1, l_2$ and $l_3$ are, respectively, the number of the first, the second and the third type of the maximal homotopic groups in $\mathcal{C}$. In particular, a maximum genus emebdding ${\pi'}_{\lfloor \frac{\beta(G_1 \bullet \mathcal{E})}{2} \rfloor}$ of $G_1 \bullet \mathcal{E}$ determines a near-triangular embedding of $G$ in $S_{\lfloor \frac{\beta(G_1 \bullet \mathcal{E})}{2} + l_1 + l_2 + l_3 \rfloor}$. Now we perform several more handle-inceasing operations on the faces around the ends of $e$ which will, one by one, merge the 3-gons (incident with the two ends of $e$) into a bigger one and finally embed $G$ into $S_{\lfloor \frac{\beta(G)}{2} \rfloor}$.

**Remark** Now we explain why the above procedure will work. Suppose that the maximum set of parallel essential 3-cycles passing through $e$ is $\mathcal{C}$. Then cycles in $\mathcal{C}$ are classified into several more maximal groups such that cycles in each group are pairwisely homotopic. Each of the first, the second and the third type of groups contains, respectively, one, two and at least three cycles. Then it may be partitioned into three possible classes:

$$\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3,$$

where each $\mathcal{C}_i$ $(1 \le i \le 3)$ contains $l_i (\ge 0)$ groups of the ith type. Then we have to make $l_1 + 2l_2 + l_3$ cuttings in total and get additional $l_2$ components such that each component is a $K_4$. Now we get $2l_1$, $4l_2$ and $2l_3$ copies of the 3-cycles in $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$, respectively. Further, we obtain $2l_1, 2l_2$ and $l_3$ copies of $e$ in the copies of the boundary cycles in each group. Notice that we also find another $l_3$ copies of the edges which are incident with $e$ (as shown in Fig.3). Let $\mathcal{E}$ be the set of those $(2l_1 + 2l_2 + 2l_3)$ copies. Then (as we have reasoned previously) each edge in $\mathcal{E}$ is on the common bondary of exact two 3-cycles in the resulting triangulation, say $G_1$. For each near-triangular embedding $\pi'_k$ of $G_1 \bullet \mathcal{E}$ into $S_k$, we split the $|\mathcal{E}|$ new vertices (pertaining to the elements in $\mathcal{E}$) and then identify the copies of 3-cycles (obtained above) pairwisely and get an embedding of $G$ into the surface $S_{k+l_1+l_2+l_3}$ such that all but one faces are triangular. Also in this embedding we get $2(l_1+2l_2+l_3)$ new 3-gons such that each 3-gon is either passing through $e$ or incident with one end of $e$ in $S_{k+l_1+l_2+l_3}$. In the case of maximum genus embedding of $G_1 \bullet \mathcal{E}$ into $S_M$, where $M = \lfloor \frac{\beta(G_1 \bullet \mathcal{E})}{2} \rfloor$, it will determine a near-triangular embedding of $G$ into $S_{M+l_1+2l_2+l_3}$ which has exact $2(l_1+2l_2+l_3+1)$ facial walks among which $2(l_1+2l_2+l_3)+1$ are 3-gons. Now we perform the handle-inceasing operation $(l_1+2l_2+l_3)$ times for the total new 3-gons around the two ends of $e$ which will embed $G$ into $S_k$ for $M+l_1+l_2+l_3 \le k \le \lfloor \frac{\beta(G)}{2} \rfloor$ such that all but one faces are triangular.

Another thing should be pointed out is that contracting the edges in $\mathcal{E}$ will not result in a multi-graph (i.e., a graph with a loop or a parallel edge) since otherwise we may conclude that the triangulation $G$ is not a simple 3-connected graph, a contradiction as desired.

Based on the above statement and the induction principle, the proof of the Temorem B is completed. □

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# 一个关于近 - 三角剖分嵌入的内插定理


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**摘要**: 一个近三角剖分嵌入是指一个图嵌入在一个曲面上, 使得至多可能有一个面不是三角面. 在本文中我们证明了如下结果：如果一个图 $G$ 在某个可定向曲面 $S_h$ 上有三角剖分嵌入，那么 $G$ 在 $S_k$ 上有一个近三角剖分嵌入，这里 $k = h, h+1, \cdots, \lfloor \frac{\beta(G)}{2} \rfloor$, 而 $\beta(G)$ 是图 $G$ 的 Betti 数.



**关键词**: 三角剖分嵌入；嵌入；可定向曲面.