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Drazin Spectrum and Weyl's Theorem for Operator Matrices

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Abstract: $A \in B(H)$ is called Drazin invertible if A has finite ascent and descent. Let $\sigma_D(A) = \{\lambda \in C : A - \lambda I \text{ is not Drazin invertible }\}$ be the Drazin spectrum. This paper shows that if $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is a 2×2 upper triangular operator matrix acting on the Hilbert space $H \oplus K$, then the passage from $\sigma_D(A) \cup \sigma_D(B)$ to $\sigma_D(M_C)$ is accomplished by removing certain open subsets of $\sigma_D(A) \cap \sigma_D(B)$ from the former, that is, there is equality

$$\sigma_D(A) \cup \sigma_D(B) = \sigma_D(M_C) \cup \mathcal{G},$$

where $\mathcal{G}$ is the union of certain holes in $\sigma_D(M_C)$ which happen to be subsets of $\sigma_D(A) \cap \sigma_D(B)$. Weyl's theorem and Browder's theorem are liable to fail for 2×2 operator matrices. By using Drazin spectrum, it also explores how Weyl's theorem, Browder's theorem, a-Weyl's theorem and a-Browder's theorem survive for 2×2 upper triangular operator matrices on the Hilbert space.

Key words: Weyl's theorem; a-Weyl's theorem; Browder's theorem; a-Browder's theorem; Drazin spectrum.
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1. Introduction

Let H and K be infinite dimensional Hilbert spaces, let $B(H, K)$ denote the set of bounded linear operators from H to K, and abbreviate $B(H, H)$ to $B(H)$. If $A \in B(H)$, write $\sigma(A)$ for the spectrum of A and $\sigma_a(A)$ for the approximate point spectrum of A, $\rho(A) = C \backslash \sigma(A)$. If $A \in B(H)$, we use $N(A)$ for the null space of A and $R(A)$ for the range of A. For $A \in B(H)$, if $R(A)$ is closed and $\dim N(A) < \infty$, we call A upper semi-Fredholm operator, and if $\dim H/R(A) < \infty$, then A is called lower semi-Fredholm operator. Let $\Phi_+(H)$ ($\Phi_-(H)$) be the set of all upper (lower) semi-Fredholm operators. A is called Fredholm operator if $\dim N(A) < \infty$ and $\dim H/R(A) < \infty$. Let A be semi-Fredholm and let $n(A) = \dim N(A)$ and $d(A) = \dim H/R(A)$, then we define the index of A by $\mathrm{ind}(A) = n(A) - d(A)$. An operator A is called Weyl if it is a Fredholm operator of index zero, and is called Browder if it is Fredholm " of finite ascent and descent". We write $\alpha(A)$ and $\beta(A)$ for the ascent and the descent for $A \in B(H)$ respectively. The essential spectrum

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$\sigma_e(A)$, the Weyl spectrum $\sigma_w(A)$ and the Browder spectrum $\sigma_b(A)$ of A are defined respectively by: $\sigma_e(A) = \{\lambda \in C : A - \lambda I \text{ is not Fredholm}\}$, $\sigma_w(A) = \{\lambda \in C : A - \lambda I \text{ is not Weyl}\}$ and $\sigma_b(A) = \{\lambda \in C : A - \lambda I \text{ is not Browder}\}$.

Following [1, Difinition 4.1] we say that $A \in B(H)$ is Drazin invertible (with a finite index) if there exist $B, U \in B(H)$ such that U is nilpotent and

$$AB = BA, \quad BAB = B, \quad ABA = A + U.$$

Recall that the concept of Drazin invertibility was originally introduced by Drazin in [2] where elements of an associative semigroup satisfying an equivalent relation were called pseudo-invertible. It is well known that A is Drazin invertible if and only if it has finite ascent and descent, which is also equivalent to the fact that $A = A_1 \oplus A_2$, where A_1 is invertible and A_2 nipotent (see [3, Proposition A] and [4, Corollary 2.2]). It is also well known that A is Drazin invertible if and only if A^* is Drazin invertible, where A^* is the conjugate of A. The Drazin spectrum of A is defined by:

$$\sigma_D(A) = \{\ \lambda \in C : \quad A - \lambda I \text{ is not Drazin invertible }\}.$$

If G is a compact subset of C, write *int* G for the interior points of G; *iso* G for the isolated points of G; *acc* G for the accumulation points of G; and $\partial\ G$ for the topological boundary of G. When $A \in B(H)$ and $B \in B(K)$ are given we denote by M_C an operator acting on $H \oplus K$ of the form $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, where $C \in B(K, H)$.

In Section 2, we will characterize the Drazin spectrum of M_C. Our result is: For a given pair (A, B) of operators, there is equality, for every $C \in B(K, H)$, $\sigma_D(A) \cup \sigma_D(B) = \sigma_D(M_C) \cup \mathcal{G}$, where $\mathcal{G}$ is the union of certain holes in $\sigma_D(M_C)$ which happen to be subsets of $\sigma_D(A) \cap \sigma_D(B)$.

In Section 3, we will use Drazin spectrum to study Weyl's theorem. Our result is: If $\sigma_D(A) \cap \sigma_D(B)$ has no interior points and if A is an isoloid operator for which Weyl's theorem holds, then for every $C \in B(K, H)$, Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Longrightarrow$ Weyl's theorem holds for $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.

2. Drazin Spectrum for 2×2 upper triangular operator matrices

Lemma 2.1 *Suppose $A \in B(H)$ and $B \in B(K)$. If both A and B are Drazin invertible, then for every $C \in B(K, H)$, $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is Drazin invertible. Hence for every $C \in B(K, H)$, $\sigma_D(M_C) \subseteq \sigma_D(A) \cup \sigma_D(B)$.*

Proof Suppose $\alpha(A) = \beta(A) = p$ and $\alpha(B) = \beta(B) = q$. Let $n = \max\{p, q\}$.

1) First we will prove that for any $C \in B(K, H)$, $\alpha(M_C) < \infty$. If we have $N(M_C^{2n+1}) = N(M_C^{2n})$, we get the result. So we only need to prove $N(M_C^{2n+1}) \subseteq N(M_C^{2n})$.

If $u_0 \in N(M_C^{2n+1})$ and $u_0 = (x_0, y_0)$, then:

$$0 = M_C^{2n+1}(x_0, y_0) = (A^{2n+1}x_0 + A^{2n}Cy_0 + A^{2n-1}CBy_0 + \cdots + A^nCB^ny_0 + \cdots + CB^{2n}y_0, B^{2n+1}y_0).$$

It follows that $B^{2n+1}y_0=0$ and

$$A^{2n+1}x_0+A^{2n}Cy_0+A^{2n-1}CBy_0+\cdots+A^nCB^ny_0+\cdots+CB^{2n}y_0=0.$$

Then $y_0\in N(B^{2n+1})=N(B^n)$ and hence

$$A^{2n+1}x_0+A^{2n}Cy_0+A^{2n-1}CBy_0+\cdots+A^{n+1}CB^{n-1}y_0=0,$$

which means that $A^{n+1}[A^nx_0+A^{n-1}Cy_0+A^{n-2}CBy_0+\cdots+CB^{n-1}y_0]=0$, and hence

$$A^nx_0+A^{n-1}Cy_0+A^{n-2}CBy_0+\cdots+CB^{n-1}y_0\in N(A^{n+1})=N(A^n).$$

Then $A^{2n}x_0+A^{2n-1}Cy_0+A^{2n-2}CBy_0+\cdots+A^nCB^{n-1}y_0=0$.

Now we get that

$$(A^{2n}x_0+A^{2n-1}Cy_0+\cdots+A^nCB^{n-1}y_0+A^{n-1}CB^ny_0+\cdots+CB^{2n-1}y_0,B^{2n}y_0)=0,$$

that is, $M_C^{2n}u_0=0$ and hence $u_0\in N(M_C^{2n})$. Then $N(M_C^{2n+1})=N(M_C^{2n})$, and hence M_C has finite ascent.

2) Secondly, we will prove that for any $C\in B(K,H)$, M_C has finite descent. We will prove that $R(M_C^{2n})=R(M_C^{2n+1})$, so we need to prove that $R(M_C^{2n})\subseteq R(M_C^{2n+1})$.

For any $u_0\in R(M_C^{2n})$, there exist $x\in H$ and $y\in K$ such that $u_0=M_C^{2n}(x,y)$, that is,

$$u_0=(A^{2n}x+A^{2n-1}Cy+A^{2n-1}CBy+\cdots+CB^{2n-1}y,\ B^{2n}y).$$

By $R(B^{2n})=R(B^{2n+1})$, there exists $y_0\in K$ such that $B^{2n}y=B^{2n+1}y_0$, then $y-By_0\in N(B^{2n})=N(B^n)$. Suppose $y=By_0+y_1$, where $y_1\in N(B^n)$. Then

$$\begin{aligned}u_0=&(A^{2n}x+A^{2n-1}CBy_0+A^{2n-1}Cy_1+\cdots+A^nCB^ny_0+\\&A^nCB^{n-1}y_1+A^{n-1}CB^{n+1}y_0+A^{n-2}CB^{n+2}y_0+\cdots+CB^{2n}y_0,\ B^{2n+1}y_0)\\=&([A^{2n}x+A^{2n-1}Cy_1+\cdots+A^nCB^{n-1}y_1-A^{2n}Cy_0]+A^{2n}Cy_0+\\&A^{2n-1}CBy_0+\cdots+CB^{2n}y_0,\ B^{2n+1}y_0),\end{aligned}$$

$$\begin{aligned}&A^{2n}x+A^{2n-1}Cy_1+A^{2n-2}CBy_1+\cdots+A^nCB^{n-1}y_1-A^{2n}Cy_0\\&\quad=A^n(A^nx+A^{n-1}Cy_1+A^{n-2}CBy_1+\cdots+CB^{n-1}y_1-A^nCy_0)\\&\quad\in R(A^n)=R(A^{2n+1}).\end{aligned}$$

Then there exists $x_0\in H$ such that

$$A^{2n}x+A^{2n-1}Cy_1+A^{2n-2}CBy_1+\cdots+A^nCB^{n-1}y_1-A^{2n}Cy_0=A^{2n+1}x_0.$$

Thus

$$\begin{aligned}u_0=&(A^{2n+1}x_0+A^{2n}Cy_0+A^{2n-1}CBy_0+\cdots+CB^{2n}y_0,B^{2n+1}y_0)\\=&M_C^{2n+1}(x_0,y_0)\in R(M_C^{2n+1}).\end{aligned}$$

So $R(M_C^{2n}) = R(M_C^{2n+1})$ and hence M_C has finite descent. The proof is completed. $\square$

Lemma 2.2 *For a given pair (A, B) of operators, if M_C is Drazin invertible for some $C \in B(K, H)$, then:*

(a) $\alpha(A) < \infty$ and $\beta(A^*) < \infty$;

(b) $\beta(B) < \infty$ and $\alpha(B^*) < \infty$.

Proof Without loss of generality, we suppose that $0 \in \sigma(M_C)$. Suppose $\alpha(M_C) = \beta(M_C) = n < \infty$, then $\alpha(M_C^*) = \beta(M_C^*) = n$. Since $N(A^n) \oplus \{0\} \subseteq N(M_C^n)$, we get $\alpha(A) < \infty$. In order to prove $\beta(A^*) < \infty$, we only need to prove that $R(A^{*n}) = R(A^{*n+1})$. So we only need to prove that $R(A^{*n}) \subseteq R(A^{*n+1})$. For any $u \in R(A^{*n})$, let $u = A^{*n}x$, then $M_C^{*n}(x, 0) \in R(M_C^{*n}) = R(M_C^{*n+1})$. Thus there exists $(x_0,\ y_0) \in H \oplus K$ such that $(A^{*n}x,\ B^{*n-1}C^*x + + \cdots + C^*A^{*n-1}x) = M_C^{*n+1}(x_0, y_0) = (A^{*n+1}x_0,\ B^{*n+1}y_0 + B^{*n}C^*x_0 + + \cdots + C^*A^{*n}x_0)$, then $u = A^{*n}x = A^{*n+1}x_0 \in R(A^{*n+1})$. Hence $\beta(A^*) < \infty$. By the same way, we can prove that $\beta(B) < \infty$ and $\alpha(B^*) < \infty$. $\square$

Lemma 2.3 *For a given pair (A, B) of operators, if M_C is Drazin invertible for some $C \in B(K, H)$, then A is Drazin invertible if and only if B is Drazin invertible.*

Proof Suppose that A is Drazin invertible. Then there exists $\varepsilon > 0$ such that $A - \lambda I$ and $M_C - \lambda I$ is invertible if $0 < |\lambda| < \varepsilon$. Thus we get that $B - \lambda I$ is invertible if $0 < |\lambda| < \varepsilon$. [5, P332, Theorem 10.5] asserts that B is Drazin invertible because $\beta(B) < \infty$. Conversely, if B is Drazin invertible, similarly, we know that A^* is Drazin invertible and hence A is Drazin invertible. $\square$

Remark 2.4 In Lemma 2.1, for every $C \in B(K, H)$, we have $\sigma_D(M_C) \subseteq \sigma_D(A) \cup \sigma_D(B)$. Sometimes, this inclusion is proper for given A and B. For example, let $A, B, C \in B(\ell_2)$ be defined by

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots),$$

$$B(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots),$$

$$C(x_1, x_2, x_3, \cdots) = (x_1, 0, 0, \cdots).$$

Then $\sigma(A) = \sigma(B) = \sigma_D(A) = \sigma_D(B) = \{\ \lambda \in C : |\lambda| \leq 1\ \}$. Since M_C is a unitary operator, then $\sigma_D(M_C) \subseteq \{\ \lambda \in C : |\lambda| = 1\ \}$. Thus $\sigma_D(M_C)$ is proper subset in $\sigma_D(A) \cup \sigma_D(B)$.

The following is our main theorem in this section. It says that the passage from $\sigma_D(A) \cup \sigma_D(B)$ to $\sigma_D(M_C)$ is accomplished by removing certain open subsets of $\sigma_D(A) \cap \sigma_D(B)$ from the former.

Theorem 2.5 *For a given pair (A, B) of operators there is equality, for every $C \in B(K, H)$,*

$$\sigma_D(A) \cup \sigma_D(B) = \sigma_D(M_C) \cup \mathcal{G},$$

where $\mathcal{G}$ is the union of certain holes in $\sigma_D(M_C)$ which happen to be subsets of $\sigma_D(A) \cap \sigma_D(B)$.

Proof We first claim that, for every $C \in B(K,H)$,

$$(\sigma_D(A) \cup \sigma_D(B)) \backslash (\sigma_D(A) \cap \sigma_D(B)) \subseteq \sigma_D(M_C) \subseteq \sigma_D(A) \cup \sigma_D(B). \tag{1}$$

Indeed the second inclusion in (1) follows from Lemma 2.1. For the first inclusion, let $\lambda \in (\sigma_D(A) \cup \sigma_D(B)) \backslash (\sigma_D(A) \cap \sigma_D(B))$. Then $\lambda \in \sigma_D(A) \backslash \sigma_D(B)$ or $\lambda \in \sigma_D(B) \backslash \sigma_D(A)$. Lemma 2.3 asserts that $\lambda \in \sigma_D(M_C)$ for every $C \in B(K,H)$.

Next we claim that, for every $C \in B(K,H)$,

$$\eta(\sigma_D(M_C)) = \eta(\sigma_D(A) \cup \sigma_D(B)), \tag{2}$$

where ηK denote the " polynomially convex hull " of the compact set $K \subseteq C$. Since $\sigma_D(M_C) \subseteq \sigma_D(A) \cup \sigma_D(B)$ for every $C \in B(K,H)$, we need to prove that $\partial(\sigma_D(A) \cup \sigma_D(B)) \subseteq \partial\, \sigma_D(M_C)$. But since int $\sigma_D(M_C) \subseteq$ int $(\sigma_D(A) \cup \sigma_D(B))$, it suffices to show that $\partial(\sigma_D(A) \cup \sigma_D(B)) \subseteq \sigma_D(M_C)$.

Let $\rho_D^+(A) = \{\ \lambda \in C:\ \alpha(A - \lambda I) < \infty$ and $\beta(A^* - \overline{\lambda} I) < \infty\ \}$ and $\rho_D^-(B) = \{\ \lambda \in C:\ \beta(B - \lambda I) < \infty$ and $\alpha(B^* - \overline{\lambda} I) < \infty\ \}$ and let $\sigma_D^+(A) = C \backslash \rho_D^+(A)$ and $\sigma_D^-(B) = C \backslash \rho_D^-(B)$. Then there are inclusions

$$\partial(\sigma_D(A) \cup \sigma_D(B)) \subseteq \partial\ \sigma_D(A) \cup \partial \sigma_D(B) \subseteq \sigma_D^+(A) \cup \sigma_D^-(B) \subseteq \sigma_D(M_C), \tag{3}$$

where the last inclusion follows from Lemma 2.2. For the second inclusion, if there exists $\lambda_0 \in (\partial\ \sigma_D(A) \cup \partial\ \sigma_D(B)) \backslash (\sigma_D^+(A) \cup \sigma_D^-(B))$, then there are two cases to consider.

Case 1. Suppose $\lambda_0 \in \partial\ \sigma_D(A)$. Then for any neighborhood of λ_0, there exists λ such that $A - \lambda I$ is Drazin invertible. Thus for any neighborhood of λ_0, there exists λ such that $A - \lambda I$ is invertible. And hence for any neighborhood of $\overline{\lambda_0}$, there exists μ such that $A^* - \mu I$ is invertible. Since $\beta(A^* - \overline{\lambda_0} I) < \infty$, [5, P332, Theorem 10.5] tells us that $A^* - \overline{\lambda_0} I$ is Drazin invertible and hence $A - \lambda_0 I$ is Drazin invertible. It is in contradiction to the fact that $\lambda_0 \in \sigma_D(A)$.

Case 2. Suppose $\lambda_0 \in \partial\ \sigma_D(B)$. Similarly as in case 1, we induce a contradiction.

Then the second inclusion is true. Consequently, (2) asserts that the passage from $\sigma_D(M_C)$ to $\sigma_D(A) \cup \sigma_D(B)$ is the filling in certain holes in $\sigma_D(M_C)$. But since, by (1), $(\sigma_D(A) \cup \sigma_D(B)) \backslash \sigma_D(M_C)$ is contained in $\sigma_D(A) \cap \sigma_D(B)$, it follows that the filling in certain holes in $\sigma_D(M_C)$ should occur in $\sigma_D(A) \cap \sigma_D(B)$. The proof is completed. □

Corollary 2.6 *If $\sigma_D(A) \cap \sigma_D(B)$ has no interior points, then for every $C \in B(K,H)$,*

$$\sigma_D(M_C) = \sigma_D(A) \cup \sigma_D(B). \tag{4}$$

In particular if either $A \in B(H)$ or $B \in B(K)$ is a Riesz operator, then (4) holds.

Corollary 2.7 *If either A^* or B is hyponormal, then for every $C \in B(K,H)$, (4) holds.*

Let $\rho_{\sigma D}^+(A) = \sigma(A) \backslash \sigma_D^+(A)$ and $\rho_{\sigma D}^-(B) = \sigma(B) \backslash \sigma_D^-(B)$. From Theorem 2.5, we can see that the holes in $\sigma_D(M_C)$ should lie in $\rho_{\sigma D}^+(A) \cap \rho_{\sigma D}^-(B)$. Thus we have:

Corollary 2.8 *If $\rho^+_{\sigma D}(A)\cap\rho^-_{\sigma D}(B)=\emptyset$, then (4) holds for every $C\in B(K,H)$.*

Lemma 2.9 *If $\sigma(M_C)=\sigma(A)\cup\sigma(B)$ or $\sigma_w(M_C)=\sigma_w(A)\cup\sigma_w(B)$, then (4) holds.*

Proof Suppose that $\sigma_w(M_C)=\sigma_w(A)\cup\sigma_w(B)$. If $M_C-\lambda_0I$ is Drazin invertible, then there exists $\varepsilon>0$ such that $M_C-\lambda I$ is invertible and $B-\lambda I$ is surjective if $0<|\lambda-\lambda_0|<\varepsilon$. Since λ is not in $\sigma_w(M_C)=\sigma_w(A)\cup\sigma_w(B)$, it follows that $B-\lambda I$ are Weyl. Then $B-\lambda I$ are invertible if $0<|\lambda-\lambda_0|<\varepsilon$. Now we have proved that $\lambda_0\in\text{iso }\sigma(B)\cup\rho(B)$. [5, P332, Theorem 10.5] tells us that $B-\lambda_0I$ is Drazin invertible and hence $A-\lambda_0I$ is Drazin invertible. Then λ_0 is not in $\sigma_D(A)\cup\sigma_D(B)$. If $\sigma(M_C)=\sigma(A)\cup\sigma(B)$, by the same way, we can prove the result. □

Corollary 2.10 *If $\sigma_w(A)\cap\sigma_w(B)$ (or $\sigma(A)\cap\sigma(B)$) has no interior points, then (4) holds for every $C\in B(K,H)$.*

Proof By Lemma 2.9 and Corollary 8 in [6] and Corollary 7 in [7], we get the result. □

3. Weyl's theorem for 2×2 upper triangular operator matrices

H.Weyl[8] has shown that every hermitian operator $A\in B(H)$ satisfies the equality

$$\sigma(A)\backslash\sigma_w(A)=\pi_{00}(A) \tag{5}$$

where $\pi_{00}(A)=\{\ \lambda\in\text{iso }\sigma(A):\ 0<\dim N(A-\lambda I)<\infty\ \}$. Now we say Weyl's theorem holds for $A\in B(H)$ if A satisfies the equality (5). If $\sigma_w(A)=\sigma_b(A)$, we say that Browder's theorem holds for A. Clearly, Weyl's theorem implies Browder's theorem.

Let $\Phi^-_+(H)$ be the class of all $A\in\Phi_+(H)$ with $\text{ind}(A)\le 0$, and for any $A\in B(H)$, and let

$$\sigma_{ea}(A)=\{\ \lambda\in C: A-\lambda I \text{ is not in } \Phi^-_+(H)\ \}$$

and $\sigma_{ab}(A)=\{\ \lambda\in C:\ A-\lambda I$ is not an upper semi-Fredholm operator with finite ascent$\}$. We call $\sigma_{ea}(A)$ and $\sigma_{ab}(A)$ the essential approximate point spectrum and Browder essential approximate point spectrum respectively.

Let $\pi^a_{00}(A)=\{\ \lambda\in\text{iso }\sigma_a(A),\ 0<\dim N(A-\lambda I)<\infty\ \}$. Similarly, we say that a-Weyl's theorem holds for A if there is equality $\sigma_a(A)\backslash\sigma_{ea}(A)=\pi^a_{00}(A)$, and that a-Browder's theorem holds for A if there is equality $\sigma_{ea}(A)=\sigma_{ab}(A)$.

Weyl's theorem may or may not hold for a direct sum of operators for which Weyl's theorem holds. Thus Weyl's theorem may fail for upper triangular operator matrices. So does a-Weyl's theorem. Weyl's theorem for upper triangular operator matrices is more delicate in comparison with the diagonal matrices. In this section, we consider this question: If Weyl's (a-Weyl's) theorem holds for $\begin{pmatrix} A & 0\\ 0 & B\end{pmatrix}$, when does it hold for $\begin{pmatrix} A & C\\ 0 & B\end{pmatrix}$? We begin with

Theorem 3.1 *If $\sigma_D(A)\cap\sigma_D(B)$ (or $\sigma(A)\cap\sigma(B)$) has no interior points, then for every $C\in B(K,H)$,*

(a) Browder's theorem holds for $\begin{pmatrix} A & 0\\ 0 & B\end{pmatrix}\Longrightarrow$ Browder's theorem holds for $\begin{pmatrix} A & C\\ 0 & B\end{pmatrix}$;

(b) a-Browder's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Longrightarrow$ *a-Browder's theorem holds for* $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.

Proof (a) Suppose $M_C - \lambda_0 I$ is Weyl. Then there exists $\varepsilon > 0$ such that $M_C - \lambda I$ is Weyl and hence $A - \lambda I$ is upper semi-Fredholm operator and $B - \lambda I$ is lower semi-Fredholm operator, and $A - \lambda I$ is Weyl if and only if $B - \lambda I$ is Weyl if $|\lambda - \lambda_0| < \varepsilon$.

Case 1. Suppose that $\lambda_0 \in \partial\sigma_D(A)$ or λ_0 is not in $\sigma_D(A)$. Then in any neighborhood of λ_0, there exists λ such that $A - \lambda I$ is Drazin invertible and hence in any neighborhood of λ_0, there exists μ such that $A - \mu I$ is invertible. Since $A - \lambda_0 I$ is upper semi-Fredholm operator, by perturbation theory of upper semi-Fredholm, it follows that $A - \lambda_0 I$ is Browder. Then $B - \lambda_0 I$ is Weyl and hence $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} - \lambda_0 I$ is Weyl. Browder's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} - \lambda_0 I$ is Browder. Thus $A - \lambda_0 I$ and $B - \lambda_0 I$ are Drazin invertible. Lemma 2.1 tells us that $M_C - \lambda_0 I$ is Drazin invertible. Since $M_C - \lambda_0 I$ is Weyl, we get that $M_C - \lambda_0 I$ is Browder.

Case 2. Suppose that $\lambda_0 \in \text{int } \sigma_D(A)$. Since $\sigma_D(A) \cap \sigma_D(B)$ has no interior points, we know that $\lambda_0 \in \partial\sigma_D(B)$ or λ_0 is not in $\sigma_D(B)$. The following proof is the same as the proof in Case 1.

Now we have proved that $\sigma_w(M_C) = \sigma_b(M_C)$ for every $C \in B(K, H)$, which means that Browder's theorem holds for M_C for every $C \in B(K, H)$.

(b) Suppose that $M_C - \lambda_0 I \in \Phi_+^-(H \oplus K)$. Then $A - \lambda_0 I \in \Phi_+(H)$.

Case 1. λ_0 is not in $\sigma_D(A)$ or $\lambda_0 \in \partial\sigma_D(A)$. Similarly to the proof in case 1 in (a), we know that $A - \lambda_0 I$ is Browder. By perturbation theory of semi-Fredholm operator, there exists $\varepsilon > 0$ such that $M_C - \lambda I \in \Phi_+^-(H \oplus K)$ with $N(M_C - \lambda I) \subseteq \cap_{n=1}^{\infty} R[(M_C - \lambda I)^n]$ and $A - \lambda I$ is invertible if $0 < |\lambda - \lambda_0| < \varepsilon$. Then $B - \lambda I \in \Phi_+(K)$ and hence $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} - \lambda I \in \Phi_+^-(H \oplus K)$. a-Browder's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, then $\alpha(A - \lambda I) < \infty$ and $\alpha(B - \lambda I) < \infty$ hence $\alpha(M_C - \lambda I) < \infty$. [9, Lemma 3.4] asserts that $N(M_C - \lambda I) = N(M_C - \lambda I) \cap \cap_{n=1}^{\infty} R[(M_C - \lambda I)^n] = \{0\}$ if $0 < |\lambda - \lambda_0| < \varepsilon$. Now we have that $\lambda_0 \in \text{iso } \sigma_a(M_C)$. Then M_C has single valued extension property in λ_0. [10, Theorem 15] tells us that $\alpha(M_C - \lambda_0 I) < \infty$.

Case 2. If $\lambda_0 \in \text{int}\sigma_D(A)$, then λ_0 is not in $\sigma_D(B)$ or $\lambda_0 \in \partial\sigma_D(B)$. By perturbation theory of upper semi-Fredholm, there exists $\varepsilon > 0$ such that $M_C - \lambda I \in \Phi_+(H \oplus K)$ with $N(M_C - \lambda I) \subseteq \bigcap_{n=1}^{\infty} R[(M_C - \lambda I)^n]$, $n(M_C - \lambda I)$ is constant, and $A - \lambda I \in \Phi_+(H)$ if $0 < |\lambda - \lambda_0| < \varepsilon$. There exists $\lambda_1 \in C$ such that $B - \lambda_1 I$ is invertible and $0 < |\lambda_1 - \lambda_0| < \varepsilon$. Then $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} - \lambda_1 I \in \Phi_+^-(H \oplus K)$. Similarly to the case 1 in (b), $M_C - \lambda_1 I$ is bounded below. Therefore $M_C - \lambda I$ is bounded below because $n(M_C - \lambda I)$ is constant if $0 < |\lambda - \lambda_0| < \varepsilon$. It follows that $\alpha(M_C - \lambda_0 I) < \infty$.

Then $\sigma_{ea}(M_C) = \sigma_{ab}(M_C)$, which means that a-Browder's theorem holds for M_C for every $C \in B(K, H)$. $\square$

We call A is isoloid if iso $\sigma(A) \subseteq \sigma_p(A)$, where $\sigma_p(A)$ is the set of all point spectrums. And we call A approximate isoloid (abbrev. a-isoloid) if $iso\ \sigma_a(A) \subseteq \sigma_p(A)$. Clearly, a-isoloid implies isoloid.

Remark 3.2 If $\sigma_w(A) \cap \sigma_w(B)$ had no interior points, then (a) in Theorem 3.1 is also true. But Theorem 3.1 may fail for " a-Weyl's theorem " even with the additional assumption that a-Weyl's theorem holds for A and B and both A and B are a-isoloid. To see this, let $A,\ B,\ C \in B(\ell_2)$ are defined by

$$A(x_1, x_2, x_3, \cdots) = (0, x_1, 0, x_2, 0, x_3, \cdots),$$

$$B(x_1, x_2, x_3, \cdots) = (0, x_2, 0, x_4, 0, x_6, \cdots),$$

$$C(x_1, x_2, x_3, \cdots) = (0, 0, 0, 0, \frac{1}{3}x_3, 0, \frac{1}{5}x_5, \cdots).$$

Then $\sigma_a(A) = \sigma_{ea}(A) = T,\ \sigma_D(A) = D,\ \pi_{00}^a(A) = \emptyset$ and $\sigma_a(B) = \sigma_{ea}(B) = \{0, 1\},\ \sigma_D(B) = \pi_{00}^a(B) = \emptyset$, which says that a-Weyl's theorem holds for A and B, both A and B are a-isoloid, and $\sigma_D(A) \cap \sigma_D(B)$ ($\sigma_w(A) \cap \sigma_w(B)$) has no interior points. Also a straightforward calculation shows that

$$\sigma_a \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_{ea} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = T \cup \{0\}, \quad \pi_{00}^a \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \emptyset,$$

$$\sigma_a(M_C) = \sigma_{ea}(M_C) = T \cup \{0\}, \quad \pi_{00}^a(M_C) = \{0\}.$$

Then a-Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, but fails for $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$.

But for Weyl's theorem, we have:

Theorem 3.3 *If $\sigma_D(A) \cap \sigma_D(B)$ (or $\sigma_w(A) \cap \sigma_w(B)$) has no interior points and if A is an isoloid operator for which Weyl's theorem holds, then for every $C \in B(K, H)$,*

$$\textit{Weyl's theorem holds for } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Longrightarrow \textit{Weyl's theorem holds for } \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

Proof Theorem 3.1 gives that $\sigma(M_C) \backslash \sigma_w(M_C) \subseteq \pi_{00}(M_C)$. For the reverse inclusion, suppose that $\lambda_0 \in \pi_{00}(M_C)$. Then there exists $\varepsilon > 0$ such that $M_C - \lambda I$ is invertible and hence $A - \lambda I$ is bounded below and $B - \lambda I$ is surjective if $0 < |\lambda - \lambda_0| < \varepsilon$. $\sigma_D(A) \cap \sigma_D(B)$ (or $\sigma_w(A) \cap \sigma_w(B)$) has no interior points, then $\sigma_D(M_C) = \sigma_D(A) \cup \sigma_D(B)$. Since λ is not in $\sigma_D(M_C) = \sigma_D(A) \cup \sigma_D(B)$, it follows that $A - \lambda I$ and $B - \lambda I$ are Drazin invertible. Thus $A - \lambda I$ and $B - \lambda I$ are invertible, which means that $\lambda_0 \in \text{iso } \sigma \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. The following proof is same as the proof in Theorem 2.4 in [11]. $\square$

Remark 3.4 Theorem 3.3 in this paper is not compatible with Theorem 2.4 in [11]. For example:

(a) Let $A \in B(\ell_2)$ be defined by

$$A(x_1, x_2, x_3, \cdots) = (x_2, x_4, x_6, \cdots),$$

and let $B = A - 2I$. Then

(I) $\sigma_D(A) = D,\ \sigma_D(B) = \{\ \lambda \in C:\ |\lambda + 2| \leq 1\ \}$. Then $\sigma_D(A) \cap \sigma_D(B)$ has no interior points;

(II) $\sigma_e(A) = D,\ \sigma_e^-(A) = T$ and $\sigma_e(B) = \{\ \lambda \in C:\ |\lambda + 2| \leq 1\ \},\quad \sigma_e^-(B) = \{\ \lambda \in C: |\lambda + 2| = 1\ \}$, where $\sigma_e^-(A) = \{\ \lambda \in C:\ A - \lambda I$ is not lower semi-Fredholm operator $\}$. Then both $SP(A)$ and $SP(B)$ have pseudoholes;

(III) $\sigma(A) = \sigma_w(A) = D$ and $\pi_{00}(A) = \emptyset$, then A is isoloid and Weyl's theorem holds for A;

(IV) $\sigma\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_w\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = D$ and $\pi_{00}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \emptyset$, then Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

By Theorem 3.3 in this note, Weyl's theorem holds for M_C for every $C \in B(\ell_2, \ell_2)$. But using Theorem 2.4 in [11], we do not know whether Weyl's theorem holds for M_C for every $C \in B(K, H)$.

(b) Let $T_1, T_2, B \in B(\ell_2)$ are defined by

$$T_1(x_1, x_2, x_3, \cdots) = (0, x_1, 0, x_2, 0, x_3, 0, \cdots),$$

$$T_2(x_1, x_2, x_3, \cdots) = (x_2, x_4, x_6, \cdots),$$

and

$$B(x_1, x_2, x_3, \cdots) = (0, x_1, x_2, x_3, \cdots).$$

Let $A = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$.

Then (I) $\sigma_D(A) = D,\ \sigma_D(B) = D$. Then $\sigma_D(A) \cap \sigma_D(B)$ has interior points;

(II) $\sigma_e(A) = \sigma_{SF_+}(A) = \sigma_{SF_-}(A) = D,\ \ \sigma_e(B) = \sigma_{SF_+}(B) = \sigma_{SF_-}(B) = T$, then both $SP(A)$ and $SP(B)$ have no pseudoholes;

(III) $\sigma(A) = \sigma_w(A) = D,\ \pi_{00}(A) = \emptyset$. Then A is isoloid and Weyl's theorem holds for A;

(IV) $\sigma\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_w\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = D,\ \ \pi_{00}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \emptyset$. Then Weyl's theorem holds for $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

Using Theorem 2.4 in [11], we know that for every $C \in B(\ell_2, \ell_2 \oplus \ell_2)$, Weyl's theorem holds for M_C. But using Theorem 3.3 in this paper, we do not know whether Weyl's theorem holds for M_C for every $C \in B(\ell_2, \ell_2 \oplus \ell_2)$.

For a-Weyl's theorem, similarly to the prove of Theorem 3.3, we have that:

Theorem 3.5 *If $\sigma_D(A)$ (or $\sigma(A)$) has no interior points, and if A is an a-isoloid operator for which a-Weyl's theorem holds, then for every $C \in B(K, H)$,*

$$a\text{-Weyl's theorem holds for } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \Longrightarrow a\text{-Weyl's theorem holds for } \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

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Drazin 谱和算子矩阵的 Weyl 定理

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摘要: $A\in B(H)$ 称为是一个 Drazin 可逆的算子，若 A 有有限的升标和降标. 用 $\sigma_D(A)=\{\lambda\in C:\ A-\lambda I$ 不是 Drazin 可逆的 $\}$ 表示 Drazin 谱集. 本文证明了对于 Hilbert 空间上的一个 2×2 上三角算子矩阵 $M_C=\begin{pmatrix} A & C\\ 0 & B\end{pmatrix}$, 从 $\sigma_D(A)\cup\sigma_D(B)$ 到 $\sigma_D(M_C)$ 的道路需要从前面子集中移动 $\sigma_D(A)\cap\sigma_D(B)$ 中一定的开子集，即有等式:

$$\sigma_D(A)\cup\sigma_D(B)=\sigma_D(M_C)\cup\mathcal{G},$$

其中 $\mathcal{G}$ 为 $\sigma_D(M_C)$ 中一定空洞的并，并且为 $\sigma_D(A)\cap\sigma_D(B)$ 的子集. 2×2 算子矩阵不一定满足 Weyl 定理，利用 Drazin 谱，我们研究了 2×2 上三角算子矩阵的 Weyl 定理， Browder 定理， a-Weyl 定理和 a-Browder 定理.

关键词: Weyl 定理; a-Weyl 定理; Browder 定理; a-Browder 定理; Drazin 谱.