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Preimage Entropies of Semi-Flows

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Abstract: In this paper, several preimage entropies for semi-flows on compact metric spaces are introduced and studied. We prove that most of these entropies are invariant in a certain sense under conjugacy when the semi-flows under consideration are free of fixed points. The relation between these entropies is studied and an inequality relating them is given. It is also shown that most of these entropies for semi-flow are consistent with that for the time-1 mapping.

Key words: semi-flow; preimage entropy; topological conjugacy.
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1. Introduction

In the study of dynamical systems, topological entropy is an important invariant. The concept of topological entropy for a continuous map was originally introduced by Adler, Konheim, and McAndrew[1]. Later, Bowen[2] gave an equivalent definition when the space under consideration is metrizable. In the same paper, Bowen also introduced a definition of topological entropy for one parameter flow by using separated set and spanning set. However, it is not easy to show whether it is invariant under conjugacy. In [3], [4], Thomas posed a new direction to redefine topological entropy by using strongly separated set and weakly spanning set which allow reparametrizations of orbits. He clarified that the new definition is equivalent to Bowen's definition for any semi-flow without fixed points, and it is invariant in a certain sense under conjugacy. One can see some other results on entropies of flows in [5]–[8].

It is well known that the topological entropy of a mapping measures the rate at which the action of the mapping disperses points in the future. In particular, when the mapping under consideration is a homeomorphism, the topological entropy of the mapping and that of its inverse mapping are equal. However, when the mapping is not invertible, how to describe the complexity of the system by using the "inverse orbits"? In recent years, Hurley[9] and Nitecki[10] formulated and studied several entropy-like invariants–preimage entropies based on preimage structure of a mapping. For some recent results on preimage entropies of mappings, one can see [11]–[15].

In this paper, we formulate and study several preimage entropies for a semi-flow. In Section 2, we give the definitions of two types of pointwise preimage entropies , $h_p(\varphi)$ and $h_m(\varphi)$, by

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using the separated set and the spanning set. For any semi-flow without fixed points, we give their equivalent definitions by using strongly separated set and weakly spanning set. In Section 3, two types of preimage branch entropies, $h_i(\varphi)$ and $H_i(\varphi)$, are given by using preimage separated set and preimage spanning set, preimage strongly separated set and preimage weakly spanning set, respectively. In Section 4, we show that the entropies $h(\varphi)$, $h_p(\varphi)$, $h_m(\varphi)$ and $H_i(\varphi)$ are all invariant in a certain sense under conjugacy for any semi-flow free of fixed points. In Section 5, we study the relation between these entropies, and get an inequality about them. In Section 6, We show that most of these entropies for a semi-flow are consistent with that for its time-1 mapping. So, the relation between the entropies for a continuous map and for its suspension is given.

Everywhere in this paper, let $(X,\ d)$ be a compact metric space, $\varphi : X \times \mathbf{R}^+ \longrightarrow X$ be a semi-flow on X, i.e., and let φ be continuous and satisfy the following properties:

1) $\varphi_0 x = x$, $\forall x \in X$;

2) $\varphi_{s+t} x = \varphi_s \circ \varphi_t x$, $\forall s, t \in \mathbf{R}^+$, $x \in X$.

2. Pointwise preimage entropies

Let K be a compact subset of X, and $t \in \mathbf{R}^+$, $\varepsilon > 0$.

A subset $F \subset X$ is said to be a $(\varphi, t, \varepsilon)$-spanning set of K, if for any $x \in K$, there exists $y \in F$ such that $d(\varphi_s x, \varphi_s y) \leq \varepsilon$, $s \in [0, t]$. Let $r(\varphi, t, \varepsilon, K)$ denote the smallest cardinality of any $(\varphi, t, \varepsilon)$-spanning set of K.

A subset $E \subset K$ is said to be a $(\varphi, t, \varepsilon)$-separated set of K, if for $x, y \in E, x \neq y$, there exists $s \in [0, t]$ such that $d(\varphi_s x, \varphi_s y) > \varepsilon$. Let $s(\varphi, t, \varepsilon, K)$ denote the largest cardinality of any $(\varphi, t, \varepsilon)$-separated set of K.

It is easy to prove that (similar to the proof for the mapping in [16]) for any $0 < \varepsilon_1 < \varepsilon_2$, we have $r(\varphi, t, \varepsilon_1, K) \geq r(\varphi, t, \varepsilon_2, K)$, $s(\varphi, t, \varepsilon_1, K) \geq s(\varphi, t, \varepsilon_2, K)$, and for any $\varepsilon > 0$, we have $r(\varphi, t, \varepsilon, K) \leq s(\varphi, t, \varepsilon, K) \leq r(\varphi, t, \frac{\varepsilon}{2}, K)$.

The topological entropy of φ is defined by

$$h(\varphi) = \lim_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \log s(\varphi, t, \varepsilon, X) = \lim_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \log r(\varphi, t, \varepsilon, X).$$

See [2] and [3] for more details. Now we give the following definitions.

Definition 2.1 *The two types of pointwise preimage entropies of φ are defined by*

$$\begin{aligned} h_p(\varphi) &= \sup_{x \in X} \lim_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \log s(\varphi, t, \varepsilon, \varphi_t^{-1}(x)) \\ &= \sup_{x \in X} \lim_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \log r(\varphi, t, \varepsilon, \varphi_t^{-1}(x)), \\ h_m(\varphi) &= \lim_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \log(\sup_{x \in X} s(\varphi, t, \varepsilon, \varphi_t^{-1}(x))) \\ &= \lim_{\varepsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \log(\sup_{x \in X} r(\varphi, t, \varepsilon, \varphi_t^{-1}(x))). \end{aligned}$$

Clearly, $h_p(\varphi) \le h_m(\varphi) \le h(\varphi)$. *In particular, if for any* $t \in \mathbf{R}^+$, φ_t *is a homeomorphism, then* $h_p(\varphi) = h_m(\varphi) = 0$.

A reparametrization of interval $[0,\ t]$ is an orientation-preserving homeomorphism from $[0,\ t]$ onto its image fixing the origin. Denote by $\mathrm{Rep}[0,\ t]$ the set of all reparametrizations of interval $[0,\ t]$.

A subset $F \subset X$ is said to be a $(\varphi, t, \varepsilon)$-weakly spanning set of K, if for any $x \in K$, there exist $y \in F$ and $\alpha \in \mathrm{Rep}[0,\ t]$ such that $d(\varphi_{\alpha(s)}x, \varphi_s y) \le \varepsilon,\ s \in [0,t]$. Let $R(\varphi,t,\varepsilon,K)$ denote the smallest cardinality of any (φ,t,ε)-weakly spanning set of K.

A subset $E \subset K$ is said to be a (φ,t,ε)-strongly separated set of K, if for any $x, y \in E, x \ne y$ and for any $\alpha, \beta \in \mathrm{Rep}[0,\ t]$, $d(\varphi_{\alpha(s)}x, \varphi_s y) > \varepsilon$ for some $s \in [0,t]$, or $d(\varphi_s x, \varphi_{\beta(s)} y) > \varepsilon$ for some $s \in [0,t]$. Let $S(\varphi,t,\varepsilon,K)$ denote the largest cardinality of any (φ,t,ε)-strongly separated set of K.

A subset $G \subset K$ is said to be a (φ,t,ε)-tracing set, if for any $x \in K$, there exists $y \in G$ which (φ,t,ε)-tracing x, i.e., there is a $\alpha \in \mathrm{Rep}[0,t]$ such that $d(\varphi_s x, \varphi_{\alpha(s)} y) \le \varepsilon,\ s \in [0,t]$. Let $T(\varphi,t,\varepsilon,K)$ denote the smallest cardinality of any (φ,t,ε)-tracing set of K.

By using weakly spanning set, strongly separated set and tracing set, we will give the equivalent definitions of the pointwise preimage entropies for fixed point free semi-flows.

Similar to Lemma 1.2 of [3] we have the following lemma.

Lemma 2.2 *Let* φ *be a semi-flow without fixed points. For any* $\lambda > 0$, *there exists* $\varepsilon > 0$ *such that for any* $t \in \mathbf{R}^+$, $x, y \in X$ *and* $\alpha \in \mathrm{Rep}[0,\ t]$, *if* $d(\varphi_{\alpha(s)}x,\ \varphi_s y) \le \varepsilon$, $\forall s \in [0,t]$, *then*

1) $|\alpha(s) - s| < \lambda$ *for* $s < 1,\ s \in [0,t]$;

2) $|\alpha(s) - s| < s\lambda$ *for* $s \ge 1,\ s \in [0,t]$.

Lemma 2.3 *Let* φ *be a semi-flow without fixed points. Let* K *be a compact subset of* X *and* $t \ge 1$.

1) For any $\varepsilon > 0$, *we have* $R(\varphi,t,\varepsilon,K) \le S(\varphi,t,\varepsilon,K)$. *For any* $\lambda > 0$, *there exists* $\varepsilon' > 0$ *such that* $S(\varphi,(1-\lambda)t,\varepsilon,K) \le R(\varphi,t,\frac{\varepsilon}{2},K)$ *for any* $0 < \varepsilon \le \varepsilon'$.

2) For any$\lambda > 0$, *there exists* $\varepsilon' > 0$ *such that* $T(\varphi,(1-\lambda)t,\varepsilon,K) \le R(\varphi,t,\varepsilon,K)$, $R(\varphi,(1-\lambda)t,\varepsilon,K) \le T(\varphi,t,\varepsilon,K)$ *for any* $0 < \varepsilon \le \varepsilon'$.

3) For any $\varepsilon > 0$, *we have* $T(\varphi,t,\varepsilon,K) \le r(\varphi,t,\varepsilon,K)$. *For any* $\varepsilon > 0$, *there exists* $0 < \delta < \frac{\varepsilon}{3}$ *such that for any* $\tau > 0,\ t > \tau$, $r(\varphi,t,\delta,K) \le 3 \cdot 3^{\frac{t}{\tau}} T(\varphi,t,\varepsilon,K)$.

Proof 1) From the definitions, for any $\varepsilon > 0$, a (φ,t,ε)-strongly separated set with the largest cardinality must be a (φ,t,ε)-weakly spanning set. So $R(\varphi,t,\varepsilon,K) \le S(\varphi,t,\varepsilon,K)$.

For any $\lambda > 0$, choose $\varepsilon' > 0$ satisfying Lemma 2.2 with respect to λ. For given $0 < \varepsilon \le \varepsilon'$, let E be a $(\varphi,(1-\lambda)t,\varepsilon)$-strongly separated set of K and F a $(\varphi,t,\frac{\varepsilon}{2})$-weakly spanning set of K. For any $x \in E$, we can choose some point $f(x) \in F$ and some $\alpha \in \mathrm{Rep}[0,t]$ such that

$$d(\varphi_{\alpha(s)}x, \varphi_s f(x)) \le \frac{\varepsilon}{2}, \quad 0 \le s \le t.$$

We claim that f defines an injective map from E to F. Therefore,

$$S(\varphi,(1-\lambda)t,\varepsilon,K)\le R(\varphi,t,\frac{\varepsilon}{2},K).$$

Proof of the claim: If for $x',x''\in E$ there exist $f(x'),f(x'')\in F$, $\alpha',\alpha''\in \mathrm{Rep}[0,t]$ such that

$$d(\varphi_{\alpha'(s)}x',\varphi_s f(x')),\ d(\varphi_{\alpha''(s)}x'',\varphi_s f(x''))\le\frac{\varepsilon}{2},\quad 0\le s\le t,$$

and $f(x')=f(x'')$, then $d(\varphi_{\alpha'(s)}x',\varphi_{\alpha''(s)}x'')\le\varepsilon$, $0\le s\le t$. By taking $u=\alpha'(s)$, we get

$$d(\varphi_u x',\varphi_{\alpha''\cdot\alpha'^{-1}(u)}x'')\le\varepsilon,\quad 0\le u\le(1-\lambda)t.$$

Since E is a $(\varphi,(1-\lambda)t,\varepsilon)$-strongly separated set, $x'=x''$.

2) For any $\lambda>0$, choose ε' satisfying Lemma 2.2 with respect to λ. For given $0<\varepsilon\le\varepsilon'$, let F be a (φ,t,ε)-weakly spanning set of K. Then for any $x\in K$, there exist $y\in F$ and $\alpha\in\mathrm{Rep}[0,t]$ such that $d(\varphi_{\alpha(s)}x,\varphi_s y)\le\varepsilon$, $0\le s\le t$. By taking $u=\alpha(s)$, we get

$$d(\varphi_u x,\varphi_{\alpha^{-1}(u)}y)\le\varepsilon,\quad 0\le u\le(1-\lambda)t.$$

This implies that F is a $(\varphi,(1-\lambda)t,\varepsilon)$-tracing set of K. Therefore,

$$T(\varphi,(1-\lambda)t,\varepsilon,K)\le R(\varphi,t,\varepsilon,K).$$

Similarly, we have $R(\varphi,(1-\lambda)t,\varepsilon,K)\le T(\varphi,t,\varepsilon,K)$.

3) From the definitions, for any $\varepsilon>0$, a (φ,t,ε)-spanning set of K must be a (φ,t,ε)-tracing set. So $T(\varphi,t,\varepsilon,K)\le r(\varphi,t,\varepsilon,K)$. Similar to the proof of Proposition 14 in [4], for any $\varepsilon>0$, there exists $0<\delta<\frac{\varepsilon}{3}$ such that for any $x\in X$, $\tau>0$ and $t>\tau$, the set of points which can be (φ,t,ε)-traced by x can be (φ,t,δ)-spanned by a set with the cardinality less than $3\cdot 3^{\frac{t}{\tau}}$. Therefore,

$$r(\varphi,t,\delta,K)\le 3\cdot 3^{\frac{t}{\tau}}T(\varphi,t,\varepsilon,K).\qquad\Box$$

Proposition 2.4 *For the semi-flow φ without fixed points, we have*

$$h(\varphi)=\lim_{\varepsilon\to0}\limsup_{t\to\infty}\frac{1}{t}\log A(\varphi,t,\varepsilon,X),$$

and

$$h_p(\varphi)=\sup_{x\in X}\lim_{\varepsilon\to0}\limsup_{t\to\infty}\frac{1}{t}\log A(\varphi,t,\varepsilon,\varphi_t^{-1}(x));$$

$$h_m(\varphi)=\lim_{\varepsilon\to0}\limsup_{t\to\infty}\frac{1}{t}\log(\sup_{x\in X}A(\varphi,t,\varepsilon,\varphi_t^{-1}(x))),$$

where $A(\cdot)=R(\cdot),S(\cdot)$ or $T(\cdot)$.

Proof We only prove the case of $h_m(\varphi)$. From (1) of Lemma 2.3, for any $\lambda>0$, there exists $\varepsilon>0$ such that

$$\sup_{x\in X}S(\varphi,(1-\lambda)t,\varepsilon,\varphi_t^{-1}(x))\le\sup_{x\in X}R(\varphi,t,\frac{\varepsilon}{2},\varphi_t^{-1}(x))\le\sup_{x\in X}S(\varphi,t,\frac{\varepsilon}{2},\varphi_t^{-1}(x)).$$

Since λ is arbitrary,

$$\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log(\sup_{x\in X}S(\varphi,t,\varepsilon,\varphi_t^{-1}(x)))=\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log(\sup_{x\in X}R(\varphi,t,\varepsilon,\varphi_t^{-1}(x))). \tag{1}$$

Similarly, from (2) of Lemma 2.3, we have

$$\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log(\sup_{x\in X}R(\varphi,t,\varepsilon,\varphi_t^{-1}(x)))=\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log(\sup_{x\in X}T(\varphi,t,\varepsilon,\varphi_t^{-1}(x))). \tag{2}$$

From (3) of Lemma 2.3, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log(\sup_{x\in X}T(\varphi,t,\varepsilon,\varphi_t^{-1}(x)))\\ &\quad\le\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log(\sup_{x\in X}r(\varphi,t,\varepsilon,\varphi_t^{-1}(x)))=h_m(\varphi).\end{aligned} \tag{3}$$

And for any $\tau>0,\ t>\tau$,

$$\frac{1}{t}\log(\sup_{x\in X}r(\varphi,t,\delta,\varphi_t^{-1}(x)))\le\frac{1}{t}\log 3+\frac{1}{\tau}\log 3+\frac{1}{t}\log(\sup_{x\in X}T(\varphi,t,\varepsilon,\varphi_t^{-1}(x))).$$

Therefore,

$$\lim_{\delta\to 0}\limsup_{t\to\infty}\frac{1}{t}\log(\sup_{x\in X}r(\varphi,t,\delta,\varphi_t^{-1}(x)))\le\frac{1}{\tau}\log 3+\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log(\sup_{x\in X}T(\varphi,t,\varepsilon,\varphi_t^{-1}(x))).$$

Since τ is arbitrary,

$$h_m(\varphi)\le\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log(\sup_{x\in X}T(\varphi,t,\varepsilon,\varphi_t^{-1}(x))). \tag{4}$$

From (1), (2), (3) and (4), we completes the proof of the case of $h_m(\varphi)$. □

3. Preimage branch entropies

In this section, we formulate two types of preimage branch entropies for semi-flows.

For $x\in X,\ t\in\mathbf{R}^+$, the t-preimage tree of x under φ is the set

$$T_t(x)=\{\mathrm{Orb}_{[0,t]}(z)\mid z\in\varphi_t^{-1}(x)\},$$

where $\mathrm{Orb}_{[0,t]}(z):=\{\varphi_s z\mid s\in[0,t]\}$ is said to be a branch of $T_t(x)$.

Let $x,x'\in X,\ t\in\mathbf{R}^+$. The preimage trees $T_t(x)$ and $T_t(x')$ are said to be (t,ε)-adjacent, if for any branch $\beta=\mathrm{Orb}_{[0,t]}(z)$ of $T_t(x)$, there exists a branch $\beta'=\mathrm{Orb}_{[0,t]}(z')$ of $T_t(x')$ such that $d(\varphi_s z,\varphi_s z')\le\varepsilon,\ s\in[0,t]$ and for any branch $\beta'=\mathrm{Orb}_{[0,t]}(y')$ of $T_t(x')$, there exists a branch $\beta=\mathrm{Orb}_{[0,t]}(y)$ of $T_t(x)$ such that $d(\varphi_s y,\varphi_s y')\le\varepsilon,\ s\in[0,t]$.

A set $F\subset X$ is said to be a (φ,t,ε)-preimage spanning set of X, if for any $x\in X$, there exists $y\in F$ such that $T_t(x)$ and $T_t(y)$ are (t,ε)-adjacent. Let $r_i(\varphi,t,\varepsilon,X)$ denote the smallest cardinality of any (φ,t,ε)-preimage spanning set of X.

A set $E\subset X$ is said to be a (φ,t,ε)-preimage separated set of X, if for any $x,y\in E, x\neq y$, $T_t(x)$ and $T_t(y)$ are not (t,ε)-adjacent. Let $s_i(\varphi,t,\varepsilon,X)$ denote the largest cardinality of any (φ,t,ε)-preimage separated set of X.

One can see that for any $0<\varepsilon_1<\varepsilon_2$, $r_i(\varphi,t,\varepsilon_1,X)\geq r_i(\varphi,t,\varepsilon_2,X)$, $s_i(\varphi,t,\varepsilon_1,X)\geq s_i(\varphi,t,\varepsilon_2,X)$, and for any $\varepsilon>0$, $r_i(\varphi,t,\varepsilon,X)\leq s_i(\varphi,t,\varepsilon,X)\leq r_i(\varphi,t,\frac{\varepsilon}{2},X)$.

Definition 3.1 *The first type of preimage branch entropy of φ is defined by*

$$h_i(\varphi)=\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log s_i(\varphi,t,\varepsilon,X)=\lim_{\varepsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log r_i(\varphi,t,\varepsilon,X).$$

The preimage trees $T_t(x)$ and $T_t(x')$ are said to be (t,ε)-weakly adjacent, if for any branch $\beta=\mathrm{Orb}_{[0,t]}(z)$ of $T_t(x)$, there exist a branch $\beta'=\mathrm{Orb}_{[0,t]}(z')$ of $T_t(x')$ and $\alpha'\in\mathrm{Rep}[0,\ t]$ such that $d(\varphi_s z,\varphi_{\alpha'(s)}z')\leq\varepsilon,\ s\in[0,t]$; and for any branch $\beta'=\mathrm{Orb}_{[0,t]}(y')$ of $T_t(x')$, there exist a branch $\beta=\mathrm{Orb}_{[0,t]}(y)$ of $T_t(x)$ and $\alpha\in\mathrm{Rep}[0,\ t]$ such that $d(\varphi_{\alpha(s)}y,\varphi_s y')\leq\varepsilon,\ s\in[0,t]$.

A set $F\subset X$ is said to be a (φ,t,ε)-preimage weakly spanning set of X, if for any $x\in X$, there exists $y\in F$ such that $T_t(x)$ and $T_t(y)$ are (t,ε)-weakly adjacent. Let $R_i(\varphi,t,\varepsilon,X)$ denote the smallest cardinality of any (φ,t,ε)-preimage weakly spanning set of X.

A set $E\subset X$ is said to be a (φ,t,ε)-preimage strongly separated set of X, if for any $x,y\in E, x\neq y$, $T_t(x)$ and $T_t(y)$ are not (t,ε)-weakly adjacent. Let $S_i(\varphi,t,\varepsilon,X)$ denote the largest cardinality of any (φ,t,ε)-preimage strongly separated set of X.

One can see that for any $0<\varepsilon_1<\varepsilon_2$,

$$R_i(\varphi,t,\varepsilon_1,X)\geq R_i(\varphi,t,\varepsilon_2,X),\ \ S_i(\varphi,t,\varepsilon_1,X)\geq S_i(\varphi,t,\varepsilon_2,X).$$

Proposition 3.2 *Let φ be a semi-flow without fixed points, $t\geq 1$, then we have:*

1) For any $\varepsilon>0$, $R_i(\varphi,t,\varepsilon,X)\leq S_i(\varphi,t,\varepsilon,X)$.

2) For any $\lambda>0$, there exists $\varepsilon'>0$ such that $S_i(\varphi,(1-\lambda)t,\varepsilon,X)\leq R_i(\varphi,t,\frac{\varepsilon}{2},X)$ for any $0<\varepsilon\leq\varepsilon'$.

Proof From the definitions, for any $\varepsilon>0$, a (φ,t,ε)-preimage strongly separated set with the largest cardinality must be a (φ,t,ε)-preimage weakly spanning set. So (1) is established.

For any $\lambda>0$, choose ε' satisfying Lemma 2.2 with respect to λ'. For given $0<\varepsilon\leq\varepsilon'$, let E be a $(\varphi,(1-\lambda)t,\varepsilon)$-preimage strongly separated set of X and let F be a $(\varphi,t,\frac{\varepsilon}{2})$-weakly spanning set of K. For any $x\in E$, we can choose some point $f(x)\in F$ such that $T_t(x)$ and $T_t(f(x))$ are $(t,\frac{\varepsilon}{2})$-weakly adjacent. We claim that f defines a injective map from E to F. Therefore, (2) is established.

Proof of the claim: In fact, if we assume that for some $x,x'\in E$ we have $f(x)=f(x'):=y$, then $T_t(y)$ and $T_t(x)$ are $(t,\frac{\varepsilon}{2})$-weakly adjacent, and so does $T_t(y)$ and $T_t(x')$. So for any $z\in\varphi_t^{-1}(x)$, there exist $y'\in\varphi_t^{-1}(y)$ and $\alpha\in\mathrm{Rep}[0,\ t]$ such that $d(\varphi_s z,\ \varphi_{\alpha(s)}y')\leq\frac{\varepsilon}{2}$, $\forall s\in[0,t]$. And for y', there exist $z'\in\varphi_t^{-1}(x')$ and $\alpha'\in\mathrm{Rep}[0,\ t]$ such that

$$d(\varphi_u y',\ \varphi_{\alpha'(u)}z')\leq\frac{\varepsilon}{2},\quad\forall u\in[0,t].$$

In the above equation, let $u=\alpha(s)$. Then $d(\varphi_{\alpha(s)}y',\ \varphi_{\alpha'\circ\alpha(s)}z')\le\frac{\varepsilon}{2}, \forall s\in[0,(1-\lambda)t]$. Therefore,

$$d(\varphi_s z,\ \varphi_{\alpha'\circ\alpha(s)}z')\le d(\varphi_s z,\ \varphi_{\alpha(s)}y')+d(\varphi_{\alpha(s)}y',\ \varphi_{\alpha'\circ\alpha(s)}z')\le\varepsilon,\quad \forall s\in[0,(1-\lambda)t].$$

Similarly, for every $z'\in\varphi_t^{-1}(x')$, there must exist some point $z\in\varphi_t^{-1}(x)$ and a reparametrization in Rep$[0,\ t]$ satisfying a similar inequality. This implies that $T_t(x)$ and $T_t(x')$ are $((1-\lambda)t,\varepsilon)$-weakly adjacent. Since E is a $(\varphi,(1-\lambda)t,\varepsilon)$-preimage strongly separated set of X then $x=x'$. This completes the proof of the claim. □

From Proposition 3.2, we can give the following definition.

Definition 3.3 For any semi-flow φ without fixed points, the second type of preimage branch entropy is defined by

$$H_i(\varphi)=\lim_{\varepsilon\to0}\limsup_{t\to\infty}\frac{1}{t}\log R_i(\varphi,t,\varepsilon,X)=\lim_{\varepsilon\to0}\limsup_{t\to\infty}\frac{1}{t}\log S_i(\varphi,t,\varepsilon,X).$$

From the definitions, for any semi-flow φ without fixed points, we have $h_i(\varphi)\ge H_i(\varphi)$. by applying Lemma 2.3, we have that if for any $t\in\mathbf{R}^+$, φ_t is a homeomorphism, then $h_i(\varphi)=H_i(\varphi)=h(\varphi)$. But we do not know whether we have $h_i(\varphi)=H_i(\varphi)$ for any semi-flow without fixed points.

4. Entropies of topological conjugate semi-flows

It is said that semi-flows (X,φ) and (Y,ψ) are topological conjugate, if there is a homeomorphism $h:X\longrightarrow Y$ mapping orbits of φ onto orbits of ψ with preserved orientation. Similar to Lemmas 3.1 and 3.2 of [3], we have the following lemma.

Lemma 4.1 *If (X,φ) and (Y,ψ) are conjugate semi-flows with a conjugate homeomorphism $h:X\longrightarrow Y$ and have no fixed points, then there exists a continuous function $\sigma:X\times\mathbf{R}^+\to\mathbf{R}^+$ such that*

1) $\sigma_x(0)=0,\ \sigma_x:\mathbf{R}^+\to\mathbf{R}^+$ is a homeomorphism, $\forall x\in X$;

2) $h(\varphi_t x)=\psi_{\sigma_x(t)}h(x)$, $\forall x\in X,\ t\in\mathbf{R}^+$;

3) $\sigma_x(s+t)=\sigma_{\varphi_t x}(s)+\sigma_x(t)$, $\forall x\in X,\ t,s\in\mathbf{R}^+$;

4) There exist $m,M>0$ such that $mt\le\sigma_x(t)\le Mt$, $\forall x\in X,\ t\ge1$.

Theorem 4.2 *If (X,φ) and (Y,ψ) are conjugate semi-flows with a conjugate homeomorphism $h:X\longrightarrow Y$ and have no fixed points, and $\sigma:X\times\mathbf{R}^+\to\mathbf{R}^+$ is the continuous function in Lemma 4.1, then*

1) $mh(\psi)\le h(\varphi)\le Mh(\psi)$;

2) $mh_p(\psi)\le h_p(\varphi)\le Mh_p(\psi)$, $mh_m(\psi)\le h_m(\varphi)\le Mh_m(\psi)$;

3) $mH_i(\psi)\le H_i(\varphi)\le MH_i(\psi)$.

Proof For any $\varepsilon>0$, take $\delta>0$ such that

$$d(x,y)\le\delta\Longrightarrow d(h(x),h(y))\le\varepsilon,\quad \forall x,y\in X.$$

Let K be a compact subset of X, and let F be a (φ,t,δ)-weakly spanning set of K with the smallest cardinality. For any $y\in h(K)$ with $y=h(x)$, there exist $z\in F$ and $\alpha\in \mathrm{Rep}[0,t]$ such that

$$d(\varphi_{\alpha(s)}x,\ \varphi_s z)\le \delta,\quad s\in[0,t].$$

So

$$d(\psi_{\sigma_x(\alpha(s))}h(x),\ \psi_{\sigma_z(s)}h(z))=d(h(\varphi_{\alpha(s)}x),\ h(\varphi_s z))\le\varepsilon,\ \ s\in[0,t].$$

Let $u=\sigma_z(s),\ \beta(u)=\sigma_x(\alpha(\sigma_z^{-1}(u)))$. Then $d(\psi_{\beta(u)}y,\ \psi_u h(z))\le\varepsilon,\ u\in[0,mt]$. Therefore, $h(F)$ is a (ψ,mt,ε)-weakly spanning set of $h(K)$. This implies $R(\psi,mt,\varepsilon,h(K))\le R(\varphi,t,\delta,K)$. From the equivalent definitions of entropies (Proposition 2.4), we have

$$mh(\psi)\le h(\varphi),\ mh_p(\psi)\le h_p(\varphi),\ mh_m(\psi)\le h_m(\varphi).$$

Similarly,

$$h(\varphi)\le Mh(\psi),\ h_p(\varphi)\le Mh_p(\psi),\ h_m(\varphi)\le Mh_m(\psi).$$

This completes the proof of (1) and (2).

Let F be a (φ,t,δ)-preimage weakly spanning set of K with the smallest cardinality. For any $y\in Y$ with $y=h(x)$, take $z\in F$ such that the preimage trees $T_t(x)$ and $T_t(z)$ are (φ,t,δ)-weakly adjacent. From the above discussion, $T_t(y)$ and $T_t(h(z))$ are (ψ,mt,ε)-weakly adjacent. So $h(F)$ is a (ψ,mt,ε)-preimage weakly spanning set of of Y. Therefore,

$$R_i(\psi,mt,\varepsilon,Y)\le R_i(\varphi,t,\delta,X).$$

And then $mH_i(\psi)\le H_i(\varphi)$. Similarly, we have $H_i(\varphi)\le MH_i(\psi)$. This completes the proof of (3). $\square$

5. An inequality relating these entropies

Theorem 5.1 *Let φ be a semi-flow without fixed points. Then $h(\varphi)\le H_i(\varphi)+h_m(\varphi)$.*

Proof Let $t\ge 1$. For any $\lambda>0$, choose ε satisfying Lemma 2.2 with respect to λ and let Y be a $(\varphi,t,\frac{\varepsilon}{3})$-preimage strongly separated set of X with the largest cardinality. For any $x\in X$, Let $M(x)$ be a maximal $(\varphi,t,\frac{\varepsilon}{3})$-strongly separated set of $\varphi_t^{-1}(x)$, and write $M=\bigcup_{y\in Y}M(y)$. We claim that M is a $(\varphi,(1-\lambda+\lambda^2)t,\varepsilon)$-weak spanning set of X. So $R(\varphi,(1-\lambda+\lambda^2)t,\varepsilon,X)\le \mathrm{card}M$. From the choice of Y and M, we have

$$R(\varphi,(1-\lambda+\lambda^2)t,\varepsilon,X)\le \mathrm{card}Y\cdot\sup_{y\in Y}\{\mathrm{card}M(y)\}\le S_i(\varphi,t,\frac{\varepsilon}{3},X)\cdot\sup_{x\in X}S(\varphi,t,\frac{\varepsilon}{3},\varphi_t^{-1}(x)).$$

Therefore,

$$\begin{aligned}&\limsup_{t\longrightarrow\infty}\frac{1}{t}\log R(\varphi,(1-\lambda+\lambda^2)t,\varepsilon,X)\\ &\quad\le\limsup_{t\longrightarrow\infty}\frac{1}{t}\log S_i(\varphi,t,\frac{\varepsilon}{3},X)+\limsup_{t\longrightarrow\infty}\frac{1}{t}\log(\sup_{x\in X}S(\varphi,t,\frac{\varepsilon}{3},\varphi_t^{-1}(x))).\end{aligned}$$

Let $\varepsilon \longrightarrow 0$. And note that λ is arbitrary to get the desired inequality

$$h(\varphi) \leq H_i(\varphi) + h_m(\varphi).$$

Proof of the claim: For any $x \in X$, let $\omega = \varphi_t x$. Since Y is a maximal $(\varphi, t, \frac{\varepsilon}{3})$-preimage strongly separated set of X, it must be a $(\varphi, t, \frac{\varepsilon}{3})$-weakly spanning set of X. So either $\omega \in Y$, or there exists $y \in Y$ such that $T_t(\omega)$ and $T_t(y)$ are $(\varphi, t, \frac{\varepsilon}{3})$-weakly adjacent. In either case, there exist $z \in \varphi_t^{-1}(y)$ and $\alpha \in \mathrm{Rep}[0,t]$ such that $d(\varphi_{\alpha(s)}x, \varphi_s z) \leq \frac{\varepsilon}{3}$, $s \in [0,t]$. By the maximality of the $(\varphi, t, \frac{\varepsilon}{3})$-strongly separated set $M(y)$, there exist $z' \in M(y)$ and $\alpha' \in \mathrm{Rep}[0,t]$ such that $d(\varphi_{\alpha'(u)}z, \varphi_u z') \leq \frac{\varepsilon}{3}$, $u \in [0,t]$. By taking $\alpha'(u) = s$, we have

$$d(\varphi_s z, \varphi_{\alpha'^{-1}(s)} z') \leq \frac{\varepsilon}{3}, \quad s \in [0, (1-\lambda)t].$$

Then

$$d(\varphi_{\alpha(s)}x, \varphi_{\alpha'^{-1}(s)}z') \leq d(\varphi_{\alpha(s)}x, \varphi_s z) + d(\varphi_s z, \varphi_{\alpha'^{-1}(s)}z') \leq \frac{2\varepsilon}{3}, \quad s \in [0, (1-\lambda)t].$$

And by taking $\alpha'^{-1}(s) = v$, we have

$$d(\varphi_{\alpha \circ \alpha'(v)}x, \varphi_v z') \leq \frac{2\varepsilon}{3}, \quad v \in [0, (1-\lambda+\lambda^2)t].$$

This completes the proof of the claim. $\square$

Therefore, for any semi-flow φ without fixed point, we have

$$h_p(\varphi) \leq h_m(\varphi) \leq h(\varphi) \leq H_i(\varphi) + h_m(\varphi) \leq h_i(\varphi) + h_m(\varphi).$$

6. Entropies of semi-flow and its time-1 mapping

Before consider the relation between the entropies for a semi-flow and for its time one mapping, we first state the concepts of entropies of continuous maps[10].

Let $f: X \longrightarrow X$ be a continuous map, and K be a subset of X and $n \in \mathbf{Z}^+$, $\varepsilon > 0$.

A set $F \subset X$ is said to be an (f, n, ε)-spanning set of K, if for any $x \in K$, there exists $y \in F$ such that

$$d(f^i(x), f^i(y)) \leq \varepsilon,\ 0 \leq i \leq n-1.$$

Let $r(f, n, \varepsilon, K)$ denote the smallest cardinality of any (f, n, ε)-spanning set of K.

The topological entropy of f is defined by

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(f, n, \varepsilon, X).$$

The two types pointwise preimage entropies of f are defined by

$$h_p(f) = \sup_{x \in X} \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(f, n, \varepsilon, f^{-n}(x)),$$

and

$$h_m(f) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log(\sup_{x\in X} r(f,n,\varepsilon,f^{-n}(x))).$$

For $x\in X$, the n-preimage tree of x under f is the set

$$T_n(x) = \{\mathrm{Orb}_{[0,n]}(z) \mid z\in f^{-n}(x)\},$$

where $\mathrm{Orb}_{[0,n]}(z) := \{z, f(z), f^2(z), \dots, f^{n-1}(z)\}\in T_n(x)$ is a branch of $T_t(x)$.

For $x, x'\in X$, the preimage trees $T_n(x)$ and $T_n(x')$ are said to be (n,ε)-adjacent, if for any branch $\beta = \mathrm{Orb}_{[0,t]}(z)$ of $T_n(x)$, there exists a branch $\beta' = \mathrm{Orb}_{[0,t]}(z')$ of $T_n(x')$ such that

$$d(f^i(x), f^i(x'))\le\varepsilon,\ 0\le i\le n-1.$$

And for any branch $\beta' = \mathrm{Orb}_{[0,t]}(y')$ of $T_n(x')$, there exists a branch $\beta = \mathrm{Orb}_{[0,t]}(y)$ of $T_n(x)$ such that $d(f^i(y'), f^i(y))\le\varepsilon,\ 0\le i\le n-1$.

A set $F\subset X$ is said to be a (f,n,ε)-preimage spanning set of X, if for any $x\in X$, there exists $y\in F$ such that $T_n(x)$ and $T_n(y)$ are (n,ε)-adjacent. Let $r_i(f,n,\varepsilon,X)$ denote the smallest cardinality of any (f,n,ε)-preimage spanning set of K.

The preimage branch entropy of f is defined by

$$h_i(f) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log r_i(f,n,\varepsilon,X).$$

Theorem 6.1 *Let φ be a semi-flow, φ_1 be its time-1 mapping. Then*

1) $h(\varphi) = h(\varphi_1)$;

2) $h_p(\varphi) = h_p(\varphi_1)$, $h_m(\varphi) = h_m(\varphi_1)$;

3) $h_i(\varphi) = h_i(\varphi_1)$.

Proof Let $n\in\mathbf{Z}^+,\ t\in\mathbf{R}^+$. For any $\varepsilon>0$, choose $\delta>0$ such that

$$d(x,y)\le\delta\Longrightarrow d(\varphi_s x,\varphi_s y)\le\varepsilon,\ \ 0\le s\le 1.$$

1) Since a (φ,t,ε)-spanning set must be a $(\varphi_1,[t],\varepsilon)$-spanning set,

$$r(\varphi,t,\varepsilon,X)\ge r(\varphi_1,[t],\varepsilon,X).$$

Therefore, $h(\varphi)\ge h(\varphi_1)$. And since a (φ_1,n,δ)-spanning set must be a (φ,n,ε)-spanning set,

$$r(\varphi_1,n,\delta,X)\ge r(\varphi,n,\varepsilon,X).$$

Therefore, $h(\varphi_1)\ge h(\varphi)$.

Accordingly, we have $h(\varphi) = h(\varphi_1)$.

2) Similar to the discussion in (1), we only note that for any $x\in X$, if F is a (φ_1,n,δ)-spanning set of $\varphi_1^{-n}(x)$, then F is a (φ,n,ε)-spanning set of $\varphi_n^{-1}x$. And if F is a (φ,t,ε)-spanning set of $\varphi_t^{-1}x$, then $\varphi_{t-[t]}F$ is a $(\varphi_1,[t],\varepsilon)$-spanning set of $\varphi_1^{-[t]}(x)$.

3) Similar to the discussion in (1) and (2), we only note that a (φ_1, n, δ)-preimage spanning set of X must be a $(\varphi, n, \varepsilon)$-preimage spanning set of X. And a $(\varphi, t, \varepsilon)$-preimage spanning set of X must be a $(\varphi_1, [t], \varepsilon)$-preimage spanning set of X. $\square$

Let (X, d) be a compact metric space, $f : X \longrightarrow X$ be a continuous map, and $\theta : X \to (0, +\infty)$ be a continuous function.

The suspension semi-flow $\varphi_{(\theta)}$ of f under θ on the space

$$X_{(\theta)} = \bigcup_{0\le t\le\theta(x)} \{(x, t) \mid (x, \theta(x)) \sim (f(x), 0)\}$$

is defined for small time by $(\varphi_{(\theta)})_t(x, s) = (x, t + s),\ \ 0 \le t + s < \theta(x)$.

Note 6.2 Each suspension $\varphi_{(\theta)}$ of f under θ is conjugate to the suspension $\varphi_{(1)}$ under the constant function with value 1. In fact, the according conjugate homeomorphism is

$$h : X_{(1)} \longrightarrow X_{(\theta)}, \quad (x, t) \mapsto (x, t\theta(x)),$$

and the continuous map as in Lemma 4.1 is $\sigma : X \times \mathbf{R}^+ \to \mathbf{R}^+$, $(x, t) \mapsto t$.

Proposion 6.3 *Let $f : X \longrightarrow X$ be a continuous map, and φ be a suspension under $\theta : X \to (0, +\infty)$, then*

1) $0 < h(f) < +\infty$ implies $0 < h(\varphi) < +\infty$; $h(f) = 0$ (or $+\infty$) implies $h(\varphi) = 0$ (or $+\infty$);

2) $0 < h_a(f) < +\infty$ implies $0 < h_a(\varphi) < +\infty$; $h_a(f) = 0$(or $+\infty$) implies $h_a(\varphi) = 0$(or $+\infty$), where $a = p, m$.

In particular, if f is a homeomorphism, then

3) $0 < h_i(f) < +\infty$ implies $0 < h_i(\varphi) = H_i(\varphi) < +\infty$; $h_i(f) = 0$ (or $+\infty$) implies $h_i(\varphi) = H_i(\varphi) = 0$ (or $+\infty$).

Proof If θ is the constant function 1, and φ is the suspension of f under θ, then the time-1 mapping of φ is f itself. From Theorem 6.1, we have

$$h(f) = h(\varphi), \quad h_a(f) = h_a(\varphi), \quad a = m, p \text{ or } i.$$

1) and 2) come from Theorem 4.2 and Note 6.2, and 3) comes from Theorem 4.2, Note 6.2 and the statement at the end of Section 3. $\square$

By Theorem 6.1, we can get some information for the entropies of semi-flows from the results for the entropies of continuous maps in [10]. Fox examples, if φ is a suspension of a positively expansive mapping under constant function 1, then $h_p(\varphi) = h_m(\varphi)$; if φ is a suspension of either a positively expansive covering map or a graph mapping under constant function 1, then $h_i(\varphi) = H_i(\varphi) = 0$, and then $h(\varphi) = h_m(\varphi)$. Furthermore, [11] gave an example by the symbolic system such that $h_p(f) \neq h_m(f)$. Therefore, if φ is a suspension of such mapping f under constant function 1, then $h_p(\varphi) \neq h_m(\varphi)$.

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半流的原像熵

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摘要: 本文对紧致度量空间上的连续半流引入了几类原像熵的定义，并对它们的性质进行了研究. 证明了对于无不动点的连续半流而言，这些熵具有一定程度的拓扑共轭不变性. 对这些熵的关系进行了研究并得到了联系这些熵的不等式. 还证明了连续半流与其时刻 1 映射具有相同的拓扑熵和原像熵.

关键词: 连续半流；原像熵；拓扑共轭.