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Limiting Behavior of Delayed Sums of φ-Mixing under a Non-Identical Distribution Setup

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Abstract We present an integral test to determine the limiting behavior of delayed sums under a non-identical distribution setup for φ-mixing sequence, and deduce Chover-type laws of the iterated logarithm for them. These complement and extend the results of Vasudeva and Divanji[1] and Chen et al.[2].

Keywords stable distribution; laws of the iterated logarithm; delayed sum; integral test.

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1. Introduction and main results

Let $\{X_n, n \geq 1\}$ be a sequence of random variables with its partial sums sequence $S_n = \sum_{i=1}^n X_i$. Let $\{a_n, n \geq 1\}$ be a positive integer subsequence. Set $T_n = S_{n+a_n} - S_n$ and $\gamma_n = \log(n/a_n) + \log\log n$. The sum T_n is called a forward delayed sum[3]. Suppose X_n's has one of two distribution functions F_1 and F_2, respectively. For each $n \geq 1$, let $\tau_1(n)$ denote the number of random variables in the set $\{X_1, X_2, \ldots, X_n\}$ with distribution function F_1. Then $\tau_2(n) = n - \tau_1(n)$ denotes the number of random variables with distribution function F_2 in the set $\{X_1, X_2, \ldots, X_n\}$. Then $(\tau_1(n), \tau_2(n))$ is called the sample scheme of the sequence $\{X_n, n \geq 1\}$.

Let $U_{\tau_1(n)}$ be the sum of those $\{X_1, X_2, \ldots, X_n\}$ with distribution function F_1 and $V_{\tau_1(n)}$ be the sum of those $\{X_1, X_2, \ldots, X_n\}$ with distribution function F_2. Then one observes that $S_n = U_{\tau_1(n)} + V_{\tau_2(n)}$. One can notice in T_n there are $\tau_1(n + a_n) - \tau_1(n)$ random variables with distribution function F_1 and $n + a_n - \tau_1(n + a_n) - (n - \tau_1(n))$ random variables with distribution function F_2.

Without further detail, we always assume that $\{X_n, n \geq 1\}$ and $\{a_n, n \geq 1\}$ satisfy the above conditions.

This paper is motivated by a study by Vasudeva and Divanji[1]. They obtained the following theorem:

Theorem A *Let $\{X_n, n \geq 1\}$ be an independent sequence, and F_1 and F_2 be positive stable*

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laws with exponents α_1 and α_2 respectively with $0 < \alpha_1 \le \alpha_2 < 1$. Let $\tau_1 = [n^{\alpha_1/\alpha_2}]$ and $\{a_n, n \ge 1\}$ be a nondecreasing sequence with $0 < a_n \le n$ and a_n/n nonincreasing.

(i) If $\lim_{n\to\infty} \log(n/a_n)/\log\log n = +\infty$, then

$$\limsup_{n\to\infty} \Big(\frac{T_n}{a_n^{1/\alpha_2}}\Big)^{1/\gamma_n} = e^{1/\alpha_2} \quad a.s.$$

(ii) If $\lim_{n\to\infty} \log(n/a_n)/\log\log n = 0$, then

$$\limsup_{n\to\infty} \Big(\frac{T_n}{a_n^{1/\alpha_2}}\Big)^{1/\gamma_n} = e^{1/\alpha_1} \quad a.s.$$

(iii) If $\lim_{n\to\infty} \log(n/a_n)/\log\log n = s (\in (0, +\infty))$, then

$$\limsup_{n\to\infty} \Big(\frac{T_n}{a_n^{1/\alpha_2}}\Big)^{1/\gamma_n} = \exp\Big\{\frac{\alpha_1 s + \alpha_2}{(s+1)\alpha_1\alpha_2}\Big\} \quad a.s.$$

They only discuss the case that F_1 and F_2 are positive stable laws and $0 < \alpha_1 \le \alpha_2 < 1$. By their method, it is impossible to discuss the rest case. In this paper, combining the results of [2], we will complement and extend Theorem A in four directions, namely:

(i) We will obtain more exact results, i.e., integral test results.

(ii) We consider φ-mixing random variables instead of independent random variables.

(iii) We consider the distributions in the domain of attraction of a stable (non-Gaussian) distributions instead of the stable distributions, and the exponents of the stable laws in $(0, 2)$, not only in $(0, 1)$.

(iv) We will remove some restrictions of the sequence $\{a_n, n \ge 1\}$.

The kind of law of iterated logarithm was first obtained by Chover[4]. To this day, many author discuss the kind of law of iterated logarithm, for example, [1], [2] and [5–12].

Now we give some definitions.

Definition 1.1 *Let $\{Z_n, n \ge 1\}$ be a sequence of random variables and $\mathcal{F}_n^m = \sigma(Z_i, n \le i \le m)$. Define φ-mixing parameters as follows:*

$$\varphi(n) = \sup_{k\ge1}\{|P(B|A) - P(B)| : A \in \mathcal{F}_1^k, B \in \mathcal{F}_{k+n}^\infty, P(A) \ne 0\}.$$

If $\varphi(n) \to 0$ as $n \to \infty$, we call $\{Z_n, n \ge 1\}\varphi$-mixing.

Definition 1.2 *We call a distribution F in the domain of attraction of a stable law $\mathcal{L}$ if there exists a sequence of independent random variables $\{W_n, n \ge 1\}$ with common distribution F, and constants $A(n) \in R$ and $B(n) > 0$ such that*

$$\frac{\sum_{i=1}^n W_i - A(n)}{B(n)} \to_d \mathcal{L} \tag{1.1}$$

where $\mathcal{L}$ is one of the stable distributions with exponent $\alpha \in (0, 2)$.

By Theorem 12 of [13], (1.1) holds if and only if

$$F(-x) = \frac{C_1(x)l(x)}{x^\alpha}, \quad 1 - F(x) = \frac{C_2(x)l(x)}{x^\alpha}, \ x > 0 \tag{1.2}$$

where $C_i(x) > 0$ if $x > 0$ and $\lim_{x\to+\infty} C_i(x) = C_i, i = 1, 2,$ $C_1 + C_2 > 0$, and $l(x)$ is a non-negative slowly varying function, i.e.,

$$\lim_{x\to+\infty} \frac{l(tx)}{l(x)} = 1, \quad \forall\ t > 0.$$

And by [13], F is in the normal domain of attraction of the stable law $\mathcal{L}$ if and only if

$$F(-x) = \frac{C_1(x)}{x^\alpha}, \quad 1 - F(x) = \frac{C_2(x)}{x^\alpha}, \ x > 0 \tag{1.3}$$

where $C_i(x)$ and $C_i(x)$ are the same as those in (1.2). In this case we can take $B(n) = Cn^{1/\alpha}$ in (1.1), where $C > 0$.

First we give the limiting behavior of S_n a accurate description via integral test.

Theorem 1.1 *Let $\{X_n, n \geq 1\}$ be a φ-mixing sequence, F_1 and F_2 be in the normal domain of attractions of stable laws with exponents α_1 and α_2, respectively, $0 < \alpha_1 \leq \alpha_2 < 2$, $\tau_1 = [n^{\alpha_1/\alpha_2}]$ and $EX_n = 0$ if $E|X_n|$ exists. If $1 \leq \alpha_2 < 2$, additionally assume that*

$$\sum_{n=1}^{\infty} \varphi^{1/2}(2^n) < \infty.$$

Let $f > 0$ be a nondecreasing function. Then with probability one

$$\limsup_{n\to\infty} \frac{|S_n|}{n^{1/\alpha_2}(f(n))^{1/\alpha 1}} = \begin{cases} 0, \\ +\infty, \end{cases} \Leftrightarrow \int_1^{+\infty} \frac{\mathrm{d}\,x}{xf(x)} \begin{cases} < +\infty, \\ = +\infty. \end{cases}$$

By Theorem 1.1, we have the following corollary.

Corollary 1.1 *Let $\{X_n, n \geq 1\}$ be given as in Theorem 1.1. Then for every $\delta > 0$, we have*

$$\limsup_{n\to\infty} \frac{|S_n|}{n^{1/\alpha_2}(\log n)^{(1+\delta)/\alpha_1}} = 0 \quad a.s.$$

and

$$\limsup_{n\to\infty} \frac{|S_n|}{n^{1/\alpha_2}(\log n)^{1/\alpha_1}} = +\infty \quad a.s.$$

In particular

$$\limsup_{n\to\infty} \left| \frac{S_n}{n^{1/\alpha_2}} \right|^{1/\log\log n} = e^{1/\alpha_1} \quad a.s.$$

Remark 1.1 If $\alpha_1 = \alpha_2$, Corollary 1.1 extends the results of [4] and [2].

Theorem 1.2 *Let $\{X_n, n \geq 1\}$ be given as in Theorem 1.1 with $\varphi(1) < 1$, and $\{a_n, n \geq 1\}$ be a subsequence of positive integers with $\limsup_{n\to\infty} a_n/n < +\infty$. Let $f > 0$ be a nondecreasing function. Then with probability one*

$$\limsup_{n\to\infty} \frac{|T_n|}{n^{1/\alpha_2}(f(n))^{1/\alpha 1}} = \begin{cases} 0, \\ +\infty, \end{cases} \Leftrightarrow \int_1^{+\infty} \frac{\mathrm{d}\,x}{xf(x)} \begin{cases} < +\infty, \\ = +\infty. \end{cases}$$

Corollary 1.2 *Let $\{X_n, n \geq 1\}$ and $\{a_n, n \geq 1\}$ be given as in Theorem 1.2. Then for every $\delta > 0$, we have*

$$\limsup_{n\to\infty} \frac{|T_n|}{n^{1/\alpha_2}(\log n)^{(1+\delta)/\alpha_1}} = 0 \quad a.s.$$

and

$$\limsup_{n\to\infty} \frac{|T_n|}{n^{1/\alpha_2}(\log n)^{1/\alpha_1}} = +\infty \quad a.s.$$

In particular

$$\limsup_{n\to\infty} |\frac{T_n}{n^{1/\alpha_2}}|^{1/\log\log n} = e^{1/\alpha_1} \quad a.s.$$

Corollary 1.3 *Let $\{X_n, n \geq 1\}$ and $\{a_n, n \geq 1\}$ be given as in Theorem 1.2.*

(i) If $\lim_{n\to\infty} \log(n/a_n)/\log\log n = +\infty$, then

$$\limsup_{n\to\infty} |\frac{T_n}{a_n^{1/\alpha_2}}|^{1/\gamma_n} = e^{1/\alpha_2} \quad a.s.$$

(ii) If $\lim_{n\to\infty} \log(n/a_n)/\log\log n = 0$, then

$$\limsup_{n\to\infty} |\frac{T_n}{a_n^{1/\alpha_2}}|^{1/\gamma_n} = e^{1/\alpha_1} \quad a.s.$$

(iii) If $\lim_{n\to\infty} \log(n/a_n)/\log\log n = s(\in (0, +\infty))$, then

$$\limsup_{n\to\infty} |\frac{T_n}{a_n^{1/\alpha_2}}|^{1/\gamma_n} = \exp\{\frac{\alpha_1 s + \alpha_2}{(s+1)\alpha_1\alpha_2}\} \quad a.s.$$

Remark 1.2 Corollary 1.3 extents and complements Theorem A.

Corollary 1.4 *Let $\{X_n, n \geq 1\}$ and $\{a_n, n \geq 1\}$ be given as in Theorem 1.2. If $\alpha_1 = \alpha_2 = \alpha$, then*

$$\limsup_{n\to\infty} |\frac{T_n}{a_n^{1/\alpha}}|^{1/\gamma_n} = e^{1/\alpha} \quad a.s.$$

Remark 1.3 Corollary 1.4 extends the results of [2] and [10].

In above results, the distributions are in the normal domain of attractions of stable laws. For general case, we have:

Theorem 1.3 *Let $\{X_n, n \geq 1\}$ be a φ-mixing sequence, F_1 and F_2 be in the domain of attractions of stable laws with exponents α_1 and α_2 respectively, $0 < \alpha_1 \leq \alpha_2 < 2$, $\tau_1 = [n^{\alpha_1/\alpha_2}]$ and $EX_n = 0$ if $E|X_n|$ exists. If $1 \leq \alpha_2 < 2$, additionally assume that*

$$\sum_{n=1}^{\infty} \varphi^{1/2}(2^n) < \infty.$$

Then there exists $B(n) > 0$ such that for every $\delta > 0$,

$$\limsup_{n\to\infty} \frac{|S_n|}{B(n)(\log n)^{(1+\delta)/\alpha_1}} = 0 \quad a.s.$$

and

$$\limsup_{n\to\infty} \frac{|S_n|}{B(n)(\log n)^{1/\alpha_1}} = +\infty \quad a.s.$$

In particular

$$\limsup_{n\to\infty} \left|\frac{S_n}{B(n)}\right|^{1/\log\log n} = e^{1/\alpha_1} \quad a.s.$$

Theorem 1.4 *Let $\{X_n, n \geq 1\}$ be given as in Theorem 1.3 with $\varphi(1) < 1$ and $\{a_n, n \geq 1\}$ as in Theorem 1.2. Then there exists $B(n) > 0$ such that for every $\delta > 0$,*

$$\limsup_{n\to\infty} \frac{|T_n|}{B(n)(\log n)^{(1+\delta)/\alpha_1}} = 0 \quad a.s.$$

and

$$\limsup_{n\to\infty} \frac{|T_n|}{B(n)(\log n)^{1/\alpha_1}} = +\infty \quad a.s.$$

In particular

$$\limsup_{n\to\infty} |\frac{T_n}{B(n)}|^{1/\log\log n} = e^{1/\alpha_1} \quad a.s.$$

Corollary 1.5 *Let $\{X_n, n \geq 1\}$ and $\{a_n, n \geq 1\}$ be given as in Theorem 1.4. Then there exists $B(n) > 0$ such that*

(i) If $\lim_{n\to\infty} \log(n/a_n)/\log\log n = +\infty$, then

$$\limsup_{n\to\infty} |\frac{T_n}{B(a_n)}|^{1/\gamma_n} = e^{1/\alpha_2} \quad a.s.$$

(ii) If $\lim_{n\to\infty} \log(n/a_n)/\log\log n = 0$, then

$$\limsup_{n\to\infty} |\frac{T_n}{B(a_n)}|^{1/\gamma_n} = e^{1/\alpha_1} \quad a.s.$$

(iii) If $\lim_{n\to\infty} \log(n/a_n)/\log\log n = s(\in (0, +\infty))$, then

$$\limsup_{n\to\infty} |\frac{T_n}{B(a_n)}|^{1/\gamma_n} = \exp\{\frac{\alpha_1 s + \alpha_2}{(s+1)\alpha_1\alpha_2}\} \quad a.s.$$

2. Proofs

We need the following lemmas.

Lemma 2.1 (see Lemma 2 of [14]) *Let $\{W_n, n \geq 1\}$ and $\{Z_n, n \geq 1\}$ be two sequences of random variables. Define $\mathcal{H}_1^n = \sigma(W_i, 1 \leq i \leq n)$ and assume*

$$\varphi = \sup_{n\geq 1} \sup\{|P(B|A) - P(B)| : A \in \mathcal{H}_1^n, P(A) \neq 0, B \in \sigma(Z_n)\} < 1.$$

Then $W_n + Z_n \to 0$ a.s. and $Z_n \to 0$ in probability imply $W_n \to 0$ a.s. and $Z_n \to 0$ a.s..

In the rest of the paper, C will be used to denote various positive constant whose exact values are irrelevant. For the sake of simplicity, we denote by Y_1 the random variable with distribution function F_1.

Proof of Theorem 1.1 First assume that $\int_1^\infty \frac{\mathrm{d}\,x}{xf(x)} < \infty$. By Theorem 1.1 of [2], we have

$$\limsup_{n\to\infty} \frac{|U_{\tau_1(n)}|}{(\tau_1(n) f(\tau_1(n)))^{1/\alpha_1}} = 0 \quad \text{a.s.}$$

and

$$\limsup_{n\to\infty} \frac{|U_{\tau_2(n)}|}{(\tau_2(n)f(\tau_2(n)))^{2/\alpha_2}} = 0 \quad \text{a.s.}$$

It is easy to show that

$$\lim_{n\to\infty} \frac{(\tau_1 f(\tau_1))^{1/\alpha_1}}{n^{1/\alpha_2}(f(n))^{1/\alpha_1}} = \lim_{n\to\infty} \frac{(\tau_2 f(\tau_2))^{1/\alpha_2}}{n^{1/\alpha_2}(f(n))^{1/\alpha_1}} = 0.$$

Hence

$$\limsup_{n\to\infty} \frac{|S_n|}{n^{1/\alpha_2}(f(n))^{1/\alpha_1}} = 0 \quad \text{a.s.}$$

Now we assume that $\int_1^\infty \frac{\mathrm{d}\,x}{xf(x)} = +\infty$. If

$$\sum_{n=1}^\infty P(|X_n| \geq Mn^{1/\alpha_2}(f(n))^{1/\alpha_1}) = +\infty, \quad \forall M > 0 \tag{2.1}$$

holds, then by the Borel-Cantelli lemma for φ-mixing sequence (see Theorem 8.2.1 in [15]), we have

$$\limsup_{n\to\infty} \frac{|X_n|}{n^{1/\alpha_2}(f(n))^{1/\alpha_1}} = +\infty \quad \text{a.s.}$$

Noticing

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{|X_n|}{n^{1/\alpha_2}(f(n))^{1/\alpha_1}} \\
&\quad \leq \limsup_{n\to\infty} \frac{|S_n|}{n^{1/\alpha_2}(f(n))^{1/\alpha_1}} + \limsup_{n\to\infty} \frac{B_{n-1}(f(n-1))^{1/\alpha_1}}{n^{1/\alpha_2}(f(n))^{1/\alpha_1}} \cdot \frac{|S_{n-1}|}{(n-1)^{1/\alpha_2}(f(n-1))^{1/\alpha_1}} \\
&\quad \leq 2\limsup_{n\to\infty} \frac{|S_n|}{n^{1/\alpha_2}(f(n))^{1/\alpha_1}},
\end{aligned}$$

we have

$$\limsup_{n\to\infty} \frac{|S_n|}{n^{1/\alpha_2}(f(n))^{1/\alpha_1}} = +\infty \quad \text{a.s.}$$

Now we prove (2.1). The following inequalities

$$\begin{aligned}
&\sum_{n=1}^\infty P(|X_n| \geq Mn^{1/\alpha_2}(f(n))^{1/\alpha_1}) \\
&\quad = \sum_{k=0}^\infty \sum_{n=2^k}^{2^{k+1}-1} P(|X_n| \geq Mn^{1/\alpha_2}(f(n))^{1/\alpha_1}) \\
&\quad \geq \sum_{k=0}^\infty \sum_{n=2^k}^{2^{k+1}-1} P(|X_n| \geq M2^{(k+1)/\alpha_2}(f(2^{k+1}))^{1/\alpha_1}) \\
&\quad \geq \sum_{k=0}^\infty (\tau_1(2^{k+1}-1) - \tau_1(2^k-1))P(|Y_1| \geq M2^{(k+1)/\alpha_2}(f(2^{k+1}))^{1/\alpha_1}) \\
&\quad \geq C\sum_{k=0}^\infty (\tau_1(2^{k+1}-1) - \tau_1(2^k-1))(2^{k+1})^{-\alpha_1/\alpha_2}(f(2^{k+1}))^{-1} \\
&\quad \geq C\sum_{k=0}^\infty (f(2^{k+1}))^{-1}
\end{aligned}$$

and $\int_1^\infty \frac{d\,x}{xf(x)} = +\infty$ imply $\sum_{k=0}^\infty (f(2^{k+1}))^{-1} = +\infty$, i.e., (2.1) holds. $\square$

Proof of Theorem 1.2 Assume that $\int_1^\infty \frac{d\,x}{xf(x)} < \infty$. By Lemma 2.3 of [2], without loss of generality, we can assume that $\limsup_{x\to\infty} f(2x)/f(x) < \infty$. By Theorem 1.1, we have

$$\limsup_{n\to\infty} \frac{|S_n|}{n^{1/\alpha_2}(f(n))^{1/\alpha_1}} = 0 \quad \text{a.s.}$$

and

$$\limsup_{n\to\infty} \frac{|S_{n+a_n}|}{(n+a_n)^{1/\alpha_2}(f(n+a_n))^{1/\alpha_1}} = 0 \quad \text{a.s.}$$

Notice that $\limsup_{n\to\infty} \frac{a_n}{n} < \infty$ implies $\limsup_{n\to\infty} \frac{(n+a_n)^{1/\alpha_2}(f(n+a_n))^{1/\alpha_1}}{n^{1/\alpha_2}(f(n))^{1/\alpha_1}} < \infty$, hence

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{|T_n|}{n^{1/\alpha_2}(f(n))^{1/\alpha_1}}\\
&\le \limsup_{n\to\infty} \frac{|S_{n+a_n}|}{n^{1/\alpha_2}(f(n))^{1/\alpha_1}} + \limsup_{n\to\infty} \frac{|S_n|}{n^{1/\alpha_2}(f(n))^{1/\alpha_1}}\\
&= \limsup_{n\to\infty} \frac{(n+a_n)^{1/\alpha_2}(f(n+a_n))^{1/\alpha_1}}{n^{1/\alpha_2}(f(n))^{1/\alpha_1}} \cdot \frac{|S_{n+a_n}|}{(n+a_n)^{1/\alpha_2}(f(n+a_n))^{1/\alpha_1}}\\
&= 0 \quad \text{a.s.}
\end{aligned}$$

Now we assume that $\int_1^\infty \frac{\mathrm{d}\,x}{xf(x)} = +\infty$. Suppose

$$\limsup_{n\to\infty} \frac{|T_n|}{n^{1/\alpha_2}(f(n))^{1/\alpha_1}} = +\infty \quad \text{a.s.}$$

does not hold. Then by 0-1 law for φ-mixing sequence[15], there exists a constant $c_0 \in [0, \infty)$ such that

$$\limsup_{n\to\infty} \frac{|T_n|}{n^{1/\alpha_2}(f(n))^{1/\alpha_1}} = c_0 \quad \text{a.s.}$$

Hence

$$\lim_{n\to\infty} \frac{T_n}{n^{1/\alpha_2}(f(n)h(n))^{1/\alpha_1}} = 0 \quad \text{a.s.}$$

where $h(x)$ is determined by Lemma 2.4 of [2]. It is easy to show that

$$\frac{X_{n+1}}{n^{1/\alpha_2}(f(n)h(n))^{1/\alpha_1}} \to 0 \quad \text{in probability.}$$

Hence

$$\frac{T_n - X_{n+1}}{n^{1/\alpha_2}(f(n)h(n))^{1/\alpha_1}} \to 0 \quad \text{in probability.}$$

By Lemma 2.1, we have

$$\frac{X_{n+1}}{n^{1/\alpha_2}(f(n)h(n))^{1/\alpha_1}} \to 0 \quad \text{a.s.} \tag{2.2}$$

Hence by the Borel-Cantelli lemma for φ-mixing sequence[15], (2.2) implies that

$$\sum_{n=1}^\infty P(|X_n| \ge n^{1/\alpha_2}(f(n)g(n))^{1/\alpha_1}) < \infty.$$

But by the same argument as (2.1), we have

$$\sum_{n=1}^{\infty} P(|X_n| \geq n^{1/\alpha_2}(f(n)g(n))^{1/\alpha_1}) = \infty.$$

This leads to a contradiction. So we complete the proof. $\square$

Proof of Corollary 1.3 By Theorem 1.2, we have

$$\limsup_{n\to\infty} \frac{|T_n|}{n^{1/\alpha_2}(\log n)^{(1+\delta)/\alpha_1}} = 0 \ \text{ a.s. } \ \forall\, \delta > 0$$

and

$$\limsup_{n\to\infty} \frac{|T_n|}{n^{1/\alpha_2}(\log n)^{1/\alpha_1}} = +\infty \ \text{ a.s.}$$

Hence we have

$$P(|T_n| \geq n^{1/\alpha_2}(\log n)^{(1+\delta)/\alpha_1}, \text{ i.o.}) = 0, \quad \forall\, \delta > 0$$

and

$$P(|T_n| \geq n^{1/\alpha_2}(\log n)^{1/\alpha_1}, \text{ i.o.}) = 1.$$

So we have

$$P(\log|\frac{T_n}{a_n^{1/\alpha_2}}| \geq (1/\alpha_2)\log(n/a_n) + ((1+\delta)/\alpha_1)\log\log n, \text{ i.o.}) = 0, \quad \forall\, \delta > 0$$

and

$$P(\log|\frac{T_n}{a_n^{1/\alpha_2}}| \geq (1/\alpha_2)\log(n/a_n) + (1/\alpha_1)\log\log n, \text{ i.o.}) = 1.$$

If $\lim_{n\to\infty}\log(n/a_n)/\log\log n = \infty$, then

$$P(\log|\frac{T_n}{a_n^{1/\alpha_2}}| \geq (1+\delta_1)\gamma_n/\alpha_2, \text{ i.o.}) = 0, \quad \forall\, \delta_1 > 0$$

and

$$P(\log|\frac{T_n}{a_n^{1/\alpha_2}}| \geq (1-\delta_2)\gamma_n/\alpha_2, \text{ i.o.}) = 1, \quad \forall\, \delta_2 > 0.$$

Hence we have

$$\limsup_{n\to\infty} |\frac{T_n}{a_n^{1/\alpha_2}}|^{1/\gamma_n} = e^{1/\alpha_2} \ \text{ a.s.}$$

If $\lim_{n\to\infty}\log(n/a_n)/\log\log n = 0$, then

$$P(\log|\frac{T_n}{a_n^{1/\alpha_2}}| \geq (1+\delta_3)\gamma_n/\alpha_1, \text{ i.o.}) = 0, \quad \forall\, \delta_3 > 0$$

and

$$P(\log|\frac{T_n}{a_n^{1/\alpha_2}}| \geq (1-\delta_4)\gamma_n/\alpha_1, \text{ i.o.}) = 1, \quad \forall\, \delta_4 > 0.$$

Hence we have

$$\limsup_{n\to\infty} |\frac{T_n}{a_n^{1/\alpha_2}}|^{1/\gamma_n} = e^{1/\alpha_1} \ \text{ a.s.}$$

If $\lim_{n\to\infty}\log(n/a_n)/\log\log n = s(\in (0,\infty))$, then

$$P(\log|\frac{T_n}{a_n^{1/\alpha_2}}| \geq (\frac{\alpha_1 s + \alpha_2}{\alpha_1\alpha_2(s+1)} + \delta_5)\gamma_n, \text{ i.o.}) = 0, \quad \forall\, \delta_5 > 0$$

and

$$P(\log|\frac{T_n}{a_n^{1/\alpha_2}}| \geq (\frac{\alpha_1 s+\alpha_2}{\alpha_1\alpha_2(s+1)} - \delta_6)\gamma_n,\ \text{i.o.}) = 1,\quad \forall\ \delta_6 > 0.$$

Hence we have

$$\limsup_{n\to\infty} |\frac{T_n}{a_n^{1/\alpha_2}}|^{1/\gamma_n} = \exp(\frac{\alpha_1 s+\alpha_2}{\alpha_1\alpha_2(s+1)})\ \ \text{a.s.} \qquad \Box$$

The Proofs of Theorems 1.3, 1.4 and Corollary 1.5 are similar, so we omit them.

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