

Perturbations of G -Frames in Hilbert Spaces

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Abstract In this paper, we discuss the properties of g -frames and g -frame operators for Hilbert spaces by utilizing the method of operator theory. Furthermore, we study perturbations of g -frames, and obtain some meaningful results.

Keywords frames; g -frames; g -frame operators; perturbations.

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1. Introduction

Frames for Hilbert spaces were formally defined by Duffin and Schaeffer^[1] in 1952 to study some deep problems in nonharmonic Fourier series. Basically, Duffin and Schaeffer introduced the fundamental concept of Gabor frames for studying signal processing. Later generalized frame theory or simply g -frame theory was introduced by Sun Wenchang as a natural generalization of the frame theory in Hilbert spaces. In the paper, we consider g -frames in a Hilbert space H , and extend some of the known results about frames to g -frames.

Let H and K be Hilbert space. $\{K_i : i \in I\}$ is a sequence of subspaces of K . Where I is a subset of \mathbb{Z} and \mathbb{Z} is the set of intergers. $B(H, K_i)$ is the collection of all bounded linear operators from H into K_i , and I_H is the identity operator on H .

A sequence of $\{U_i \in B(H, K_i) : i \in I\}$ is called a g -frame for H with respect to $\{K_i : i \in I\}$, if there exist $0 < C \leq D < \infty$ such that for all $f \in H$,

$$C\|f\|^2 \leq \sum_{i \in I} \|U_i f\|^2 \leq D\|f\|^2, \quad \forall f \in H. \quad (1.1)$$

The number C and D are called a lower and upper frame bounds. A g -frame $\{U_i : i \in I\}$ is called a tight g -frame if $C = D$ and a Parseval tight g -frame if $C = D = 1$. Let $\{U_i : i \in I\}$ be a g -frame. Then the g -frame operator

$$S(f) = \sum_{i \in I} U_i^* U_i f \quad (1.2)$$

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associated with $\{U_i \in B(H, K_i) : i \in I\}$ is a bounded, invertible and positive operator mapping H onto itself. This provides the reconstruction formula

$$f = SS^{-1}(f) = S^{-1}S(f) = \sum_{i \in I} U_i^* \tilde{U}_i f = \sum_{i \in I} \tilde{U}_i^* U_i f, \quad \forall f \in H, \quad (1.3)$$

where $\tilde{U}_i = U_i S^{-1}$. The family $\{\tilde{U}_i : i \in I\}$ is also a g -frame for H , called the canonical dual g -frame of $\{U_i : i \in I\}$.

The main result of this paper deals with g -frame operators on H and perturbations of g -frames. We refer to [2] for an excellent introduction to g -frames. Our references for frames are [1, 3, 5, 6].

2. Main results

In order to prove our assertions, we first give the following lemma.

Lemma 2.1 *Let $\{U_i \in B(H, K_i) : i \in I\}$ be a family of operators. We define an operator S by $Sf = \sum_{i \in I} U_i^* U_i f, \forall f \in H$. Then the family of operators $\{U_i : i \in I\}$ is a g -frame for H with respect to $\{K_i : i \in I\}$ if and only if S is a positive and invertible operator on H .*

Proof Necessity is clear. We only need to prove sufficiency. Conversely, if the operator S is a positive and invertible operator, then for any $f \in H$, we have that $\Delta(S)\|f\|^2 \leq \langle Sf, f \rangle \leq \|S\| \cdot \|f\|^2$. Hence, the family of operators $\{U_i : i \in I\}$ is a g -frame with the frame bounds $\Delta(S)$ and $\|S\|$, where $\Delta(S) = \inf\{\|Sf\| : f \in H, \|f\| = 1\}$ denotes the minimal module of S .

Indeed, Lemma 2.1 gives a relation between g -frames and g -frame operators.

Suppose that a g -frame is Parseval tight. We have that $\langle Sf, f \rangle = \|f\|^2$ for all $f \in H$. In this case, $S = I_H$. Furthermore, we get the following assertion.

Proposition 2.2 *Let $\{U_i \in B(H, K_i) : i \in I\}$ be a g -frame for H with respect to $\{K_i : i \in I\}$ with the frame operator S , and $\{\tilde{U}_i \in B(H, K_i) : i \in I\}$ be the canonical dual g -frame of $\{U_i : i \in I\}$. Then the following assertions are equivalent.*

- (1) $S = I_H$.
- (2) $\|f\|^2 = \sum_{i \in I} \|U_i f\|^2, \forall f \in H$.
- (3) $\|f\|^2 = \sum_{i \in I} \|\tilde{U}_i f\|^2, \forall f \in H$.

Proof We easily see from (1.2) that (1) \Rightarrow (2) and (1) \Rightarrow (3) are evident.

(2) \Rightarrow (1) Since $S^{-1}f = \sum_{i \in I} S^{-1}U_i^* \tilde{U}_i f$, for any $f \in H$, we have

$$\langle S^{-1}f, f \rangle = \sum_{i \in I} \langle S^{-1}U_i^* U_i S^{-1}f, f \rangle = \sum_{i \in I} \|U_i S^{-1}f\|^2 = \langle S^{-2}f, f \rangle.$$

Hence $S^{-1} = S^{-2}$. This implies that $S = I_H$.

(3) \Rightarrow (1) Using the formula (1.3), we see that

$$\langle Sf, f \rangle = \sum_{i \in I} \langle U_i^* U_i f, f \rangle = \sum_{i \in I} \|\tilde{U}_i S f\|^2 = \langle S^2 f, f \rangle, \quad \forall f \in H.$$

Thus $S = S^2$. This implies that $S = I_H$.

The next theorem is analogous to Proposition 4.1 in [3], but the proof is different from [3].

Theorem 2.3 *Let $\{U_i \in B(H, K_i) : i \in I\}$ be a g -frame for H with respect to $\{K_i : i \in I\}$ with frame bounds C and D , and let $\{V_i \in B(H, K_i) : i \in I\}$ be a family of operators. If there exists a $0 < R < C$ such that*

$$\sum_{i \in I} \|U_i(f) - V_i(f)\|^2 \leq R\|f\|^2 \quad (2.1)$$

for all $f \in H$, then $\{V_i : i \in I\}$ is a g -frame for H with respect to $\{K_i : i \in I\}$ with frame bounds $(\sqrt{C} - \sqrt{R})$ and $2(D + R)$.

Proof For a family of operators $\{V_i\}_{i \in I}$, the operator S is defined by

$$Sf = \sum_{i \in I} V_i^* V_i f, \quad \forall f \in H. \quad (2.2)$$

By Lemma 2.1, to prove that $\{V_i\}_{i \in I}$ is a g -frame, it suffices to show that S is a positive invertible operator.

First of all, S is well defined on H . To see this, let $n < m$ be integers. Then we have

$$\begin{aligned} \left\| \sum_{i=n}^m V_i^* V_i f \right\| &= \sup_{h \in H, \|h\|=1} \left| \left\langle \sum_{i=n}^m V_i^* V_i f, h \right\rangle \right| = \sup_{\|h\|=1} \left| \sum_{i=n}^m \langle V_i f, V_i h \rangle \right| \\ &\leq \sup_{\|h\|=1} \left(\sum_{i=n}^m \|V_i f\|^2 \right)^{1/2} \cdot \left(\sum_{i=n}^m \|V_i h\|^2 \right)^{1/2} \\ &\leq \sqrt{2}(D + R)^{1/2} \left(\sum_{i=n}^m \|V_i f\|^2 \right)^{1/2}. \end{aligned}$$

Now we see from (2.1) that the series in (2.2) are convergent. Thus, S is well defined for any $f \in H$.

On the other hand, it is easy to check that for any $f \in H$,

$$\begin{aligned} \langle Sf, f \rangle^{1/2} &= \left(\sum_{i \in I} \|V_i f\|^2 \right)^{1/2} \\ &\geq \left(\sum_{i \in I} \|U_i f\|^2 \right)^{1/2} - \left(\sum_{i \in I} \|(U_i - V_i)f\|^2 \right)^{1/2} \\ &\geq (\sqrt{C} - \sqrt{R})^{1/2} \|f\|, \end{aligned}$$

and therefore,

$$\langle Sf, f \rangle \geq (\sqrt{C} - \sqrt{R})\|f\|^2, \quad \forall f \in H. \quad (2.3)$$

Hence S is positive and bounded below.

We now estimate an upper bound of S . For any $f \in H$, we have

$$\begin{aligned} \langle Sf, f \rangle^{1/2} &= \left(\sum_{i \in I} \|V_i f\|^2 \right)^{1/2} = \left(\sum_{i \in I} \|(V_i - U_i + U_i)f\|^2 \right)^{1/2} \\ &\leq \sqrt{2} \left(\sum_{i \in I} \|(V_i - U_i)f\|^2 + \sum_{i \in I} \|U_i f\|^2 \right)^{1/2} \end{aligned}$$

$$\leq \sqrt{2}(D+R)^{1/2}\|f\|,$$

therefore,

$$\langle Sf, f \rangle \leq 2(D+R)\|f\|^2, \quad \forall f \in H. \quad (2.4)$$

Combining (2.3) with (2.4), for any $f \in H$, we have

$$(\sqrt{C} - \sqrt{R})\|f\|^2 \leq \langle Sf, f \rangle \leq 2(D+R)\|f\|^2. \quad (2.5)$$

From (2.5), it is clear that S is invertible.

Theorem 2.4 Let $\{U_i \in B(H, K_i) : i \in I\}$ be a g -frame for H with respect to $\{K_i : i \in I\}$ with frame bounds C_U and D_U , and let $\{V_i \in B(H, K_i) : i \in I\}$ be a family of operators. The following are equivalent.

(1) $\{V_i \in B(H, K_i) : i \in I\}$ is a g -frame for H with respect to $\{K_i : i \in I\}$ with frame bounds C_V and D_V .

(2) There is a constant $M > 0$ so that for any $f \in H$ we have

$$\sum_{i \in I} \|(U_i - V_i)f\|^2 \leq M \min \left(\sum_{i \in I} \|U_i f\|^2, \sum_{i \in I} \|V_i f\|^2 \right). \quad (2.6)$$

Proof (1) \Rightarrow (2). For any $f \in H$, we have

$$\begin{aligned} \sum_{i \in I} \|(U_i - V_i)f\|^2 &\leq 2 \left(\sum_{i \in I} \|U_i f\|^2 + \sum_{i \in I} \|V_i f\|^2 \right) \\ &\leq 2 \left(\sum_{i \in I} \|U_i f\|^2 + D_V \|f\|^2 \right) \\ &\leq 2 \left(\sum_{i \in I} \|U_i f\|^2 + \frac{D_V}{C_U} \sum_{i \in I} \|U_i f\|^2 \right) \\ &\leq 2 \left(1 + \frac{D_V}{C_U} \right) \sum_{i \in I} \|U_i f\|^2. \end{aligned} \quad (2.7)$$

By symmetry, since we are assuming in (1) that $\{U_i \in B(H, K_i) : i \in I\}$ is also a g -frame for H with respect to $\{K_i : i \in I\}$, we have

$$\sum_{i \in I} \|(U_i - V_i)f\|^2 \leq 2 \left(1 + \frac{D_U}{C_V} \right) \sum_{i \in I} \|V_i f\|^2. \quad (2.8)$$

Combining (2.7) with (2.8), we see that (2.6) holds.

(2) \Rightarrow (1). By (2.6), for any $f \in H$ we have

$$\begin{aligned} C_U \|f\|^2 &\leq \sum_{i \in I} \|U_i f\|^2 \leq 2 \left(\sum_{i \in I} \|(U_i - V_i)f\|^2 + \sum_{i \in I} \|V_i f\|^2 \right) \\ &\leq 2 \left(M \sum_{i \in I} \|V_i f\|^2 + \sum_{i \in I} \|V_i f\|^2 \right) \\ &= 2(M+1) \sum_{i \in I} \|V_i f\|^2, \end{aligned}$$

and

$$\begin{aligned} \sum_{i \in I} \|V_i f\|^2 &\leq 2 \left(\sum_{i \in I} \|(U_i - V_i)f\|^2 + \sum_{i \in I} \|U_i f\|^2 \right) \\ &\leq 2 \left(M \sum_{i \in I} \|U_i f\|^2 + \sum_{i \in I} \|U_i f\|^2 \right) \\ &\leq 2(M+1) \sum_{i \in I} \|U_i f\|^2 \leq 2(M+1) D_U \|f\|^2. \end{aligned}$$

Therefore, we have

$$\frac{C_U}{2(M+1)} \|f\|^2 \leq \sum_{i \in I} \|V_i f\|^2 \leq 2(M+1) D_U \|f\|^2. \quad \square$$

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